A Semi-Analytic Diagonalization Tensor FEM for the Spectral Fractional Laplacian.

Shane Sawyer, sjw355@tennessee.edu
Advisor: Abner J. Salgado

University of Tennessee, Knoxville

Fall 2023 Finite Element Circus

THE UNIVERSITY OF TENNESSEE KNOXVILLE
DEPARTMENT OF MATHEMATICS

Introduction

- Recall $(-\Delta)$: $(\lambda_k, \phi_k)_{k\geq 1}$, eigenbasis of $L^2(\Omega)$.

Introduction

- Recall $(-\Delta)$: $(\lambda_k, \phi_k)_{k\geq 1}$, eigenbasis of $L^2(\Omega)$.
- We want to solve: $(-\Delta)^s u = f$ in Ω

Introduction

- Recall $(-\Delta)$: $(\lambda_k, \phi_k)_{k\geq 1}$, eigenbasis of $L^2(\Omega)$.
- We want to solve: $(-\Delta)^s u = f$ in Ω
- For $s \in (0,1)$ and any $w \in C_0^\infty(\Omega)$, $(-\Delta)^s$ defined as

$$(-\Delta)^s w(x) = \sum_{k\geq 1} \lambda_k^s w_k \phi_k(x) \quad , \quad (-\Delta)^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$$

Introduction

- Recall $(-\Delta)$: $(\lambda_k, \phi_k)_{k\geq 1}$, eigenbasis of $L^2(\Omega)$.
- We want to solve: $(-\Delta)^s u = f$ in Ω
- For $s \in (0,1)$ and any $w \in C_0^\infty(\Omega)$, $(-\Delta)^s$ defined as

$$(-\Delta)^s w(x) = \sum_{k\geq 1} \lambda_k^s w_k \phi_k(x) \quad , \quad (-\Delta)^s : \mathbb{H}^s(\Omega) \to \mathbb{H}^{-s}(\Omega)$$

- $\mathbb{H}^s(\Omega) = \left\{ w(x) = \sum_{k\geq 1} w_k \phi_k(x) \ : \ \sum_{k\geq 1} \lambda_k^s |w_k|^2 < \infty \right\}$

Extension

Extension

- Use the Caffarelli-Silvestre extension technique for bounded Ω: $\mathcal{U}(\mathbf{x}) = \mathcal{U}(x, y)$ where $x \in \Omega$ and $y \in \mathbb{R}^+$ solves

$$\begin{cases} \operatorname{div}(y^\alpha \nabla \mathcal{U}) = 0 & \text{in } \mathcal{C} \\ \mathcal{U} = 0 & \text{on } \partial_L \mathcal{C} \\ \partial_{\nu^\alpha} \mathcal{U} = d_s f & \text{on } \Omega \times \{0\} \end{cases}$$

where $\mathcal{C} = \Omega \times (0, \infty)$, $\partial_L \mathcal{C} = \partial\Omega \times [0, \infty)$, $\alpha = 1 - 2s$

Extension

- Use the Caffarelli-Silvestre extension technique for bounded Ω: $\mathcal{U}(\mathbf{x}) = \mathcal{U}(x, y)$ where $x \in \Omega$ and $y \in \mathbb{R}^+$ solves

$$\begin{cases} \operatorname{div}(y^\alpha \nabla \mathcal{U}) = 0 & \text{in } \mathcal{C} \\ \mathcal{U} = 0 & \text{on } \partial_L \mathcal{C} \\ \partial_{\nu^\alpha} \mathcal{U} = d_s f & \text{on } \Omega \times \{0\} \end{cases}$$

  where $\mathcal{C} = \Omega \times (0, \infty)$, $\partial_L \mathcal{C} = \partial\Omega \times [0, \infty)$, $\alpha = 1 - 2s$
- $\mathring{H}^1_L(y^\alpha, \mathcal{C}) = \left\{ w \in L^2(y^\alpha, \mathcal{C}) : \nabla w \in L^2(y^\alpha, \mathcal{C}), w|_{\partial_L \mathcal{C}} = 0 \right\}$

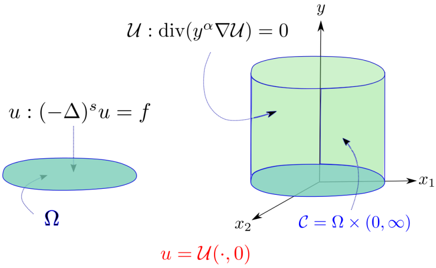
Extension

- Use the Caffarelli-Silvestre extension technique for bounded Ω: $\mathcal{U}(\mathbf{x}) = \mathcal{U}(x, y)$ where $x \in \Omega$ and $y \in \mathbb{R}^+$ solves

$$\begin{cases} \operatorname{div}(y^\alpha \nabla \mathcal{U}) = 0 & \text{in } \mathcal{C} \\ \mathcal{U} = 0 & \text{on } \partial_L \mathcal{C} \\ \partial_{\nu^\alpha} \mathcal{U} = d_s f & \text{on } \Omega \times \{0\} \end{cases}$$

where $\mathcal{C} = \Omega \times (0, \infty)$, $\partial_L \mathcal{C} = \partial\Omega \times [0, \infty)$, $\alpha = 1 - 2s$

- $\mathring{H}^1_L(y^\alpha, \mathcal{C}) = \left\{ w \in L^2(y^\alpha, \mathcal{C}) : \nabla w \in L^2(y^\alpha, \mathcal{C}), w|_{\partial_L \mathcal{C}} = 0 \right\}$



Truncation

- Cylinder must be finite: $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$

[1] Ricardo H Nochetto, Enrique Otárola, and Abner J Salgado. "A PDE approach to fractional diffusion in general domains: a priori error analysis". In: *Foundations of Computational Mathematics* 15.3 (2015), pp. 733–791.

Truncation

- Cylinder must be finite: $\mathcal{C}_\mathcal{Y} = \Omega \times (0, \mathcal{Y})$
- $\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$ defined analogously, adding $w(\cdot, \mathcal{Y}) = 0$

[1] Nochetto, Otárola, and Salgado, "A PDE approach to fractional diffusion in general domains: a priori error analysis".

Truncation

- Cylinder must be finite: $\mathcal{C}_\mathcal{Y} = \Omega \times (0, \mathcal{Y})$
- $\mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ defined analogously, adding $w(\cdot, \mathcal{Y}) = 0$
- Theorem (exponential decay)[1] For every $\mathcal{Y} \geq 1$,

$$\|\mathcal{U}\|_{\mathring{H}_L^1(y^\alpha, \Omega \times (\mathcal{Y}, \infty))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \|f\|_{\mathbb{H}^{-s}(\Omega)}$$

[1] Nochetto, Otárola, and Salgado, "A PDE approach to fractional diffusion in general domains: a priori error analysis".

Truncation

- Cylinder must be finite: $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$
- $\mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathcal{Y}})$ defined analogously, adding $w(\cdot, \mathcal{Y}) = 0$
- Theorem (exponential decay)[1] For every $\mathcal{Y} \geq 1$,

$$\|\mathcal{U}\|_{\mathring{H}_L^1(y^\alpha, \Omega \times (\mathcal{Y}, \infty))} \lesssim e^{-\sqrt{\lambda_1}\mathcal{Y}/2} \|f\|_{\mathbb{H}^{-s}(\Omega)}$$

- $\mathcal{U}_{\mathcal{Y}} \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_{\mathcal{Y}})$ solve

$$\begin{cases} \text{div } (y^\alpha \nabla \mathcal{U}_{\mathcal{Y}}) = 0 & \text{in } \mathcal{C}_{\mathcal{Y}} \\ \mathcal{U}_{\mathcal{Y}} = 0 & \text{on } \partial_L \mathcal{C}_{\mathcal{Y}} \cup \Omega \times \{\mathcal{Y}\} \\ \partial_{\nu^\alpha} \mathcal{U}_{\mathcal{Y}} = d_s f & \text{on } \Omega \times \{0\} \end{cases}$$

[1] Nochetto, Otárola, and Salgado, "A PDE approach to fractional diffusion in general domains: a priori error analysis".

Truncation

- Cylinder must be finite: $\mathcal{C}_\mathcal{Y} = \Omega \times (0, \mathcal{Y})$
- $\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$ defined analogously, adding $w(\cdot, \mathcal{Y}) = 0$
- Theorem (exponential decay)[1] For every $\mathcal{Y} \geq 1$,

$$\|\mathcal{U}\|_{\mathring{H}^1_L(y^\alpha, \Omega \times (\mathcal{Y}, \infty))} \lesssim e^{-\sqrt{\lambda_1}\,\mathcal{Y}/2} \|f\|_{\mathbb{H}^{-s}(\Omega)}$$

- $\mathcal{U}_\mathcal{Y} \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$ solve

$$\begin{cases} \text{div}\,(y^\alpha \nabla \mathcal{U}_\mathcal{Y}) = 0 & \text{in } \mathcal{C}_\mathcal{Y} \\ \mathcal{U}_\mathcal{Y} = 0 & \text{on } \partial_L \mathcal{C}_\mathcal{Y} \cup \Omega \times \{\mathcal{Y}\} \\ \partial_{\nu^\alpha} \mathcal{U}_\mathcal{Y} = d_s f & \text{on } \Omega \times \{0\} \end{cases}$$

- Theorem (exponential convergence) For every $\mathcal{Y} \geq 1$, [1]

$$\|\mathcal{U} - \mathcal{U}_\mathcal{Y}\|_{\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})} \lesssim e^{-\sqrt{\lambda_1}\,\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)}$$

[1] Nochetto, Otárola, and Salgado, "A PDE approach to fractional diffusion in general domains: a priori error analysis".

Weak Form and Discretization

Weak Form and Discretization

- $a_{\mathcal{C}_\mathcal{Y}}(V,\phi) = \int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V \cdot \nabla \phi dx' dy = d_s \int_\Omega f \phi(x',0) dx'$, $\forall \phi \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$

Weak Form and Discretization

- $a_{\mathcal{C}_\mathcal{Y}}(V,\phi) = \int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V \cdot \nabla \phi dx' dy = d_s \int_\Omega f\phi(x',0)dx'$, $\forall \phi \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$

- Decomposition of function space:

Theorem

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}) = H^1_0(\Omega) \otimes H^1_\mathcal{Y}(y^\alpha, (0, \mathcal{Y}))$$

Weak Form and Discretization

- $a_{\mathcal{C}_\mathcal{Y}}(V,\phi) = \int_{\mathcal{C}_\mathcal{Y}} y^\alpha \nabla V \cdot \nabla \phi dx' dy = d_s \int_\Omega f\phi(x',0)dx'$, $\forall \phi \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$

- Decomposition of function space:

Theorem

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}) = H^1_0(\Omega) \otimes H^1_\mathcal{Y}(y^\alpha, (0,\mathcal{Y}))$$

- Full FE space: $\mathbb{V}_{h,\mathcal{Y}} = \mathbb{V}_h \otimes \mathcal{S}_\mathcal{Y} \subset \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$

Discrete Problem

Look for solutions

$$\mathcal{U}^K_{h,\mathcal{Y}}(x'y) = \sum_{k=1}^{K} U_k(x')V_k(y)\,, \quad U_k \in \mathbb{V}_h\,,\, V_k \in \mathcal{S}_\mathcal{Y}$$

Discrete Problem

Look for solutions

$$\mathcal{U}_{h,\mathcal{Y}}^{K}(x'y) = \sum_{k=1}^{K} U_k(x')V_k(y)\,, \quad U_k \in \mathbb{V}_h\,,\; V_k \in \mathcal{S}_{\mathcal{Y}}$$

Our problem reads:

$$\int_{\mathcal{C}_{\mathcal{Y}}} y^{\alpha}\left[\nabla_{x',y}\sum_{k=1}^{K} U_k(x')V_k(y)\right]\,\nabla_{x',y}\phi(x',y)\,dx'dy = d_s\int_{\Omega} f(x')\phi(x',0)dx'\,,$$

$\forall\,\phi \in \mathbb{V}_{h,\mathcal{Y}}$

Discrete Problem

Look for solutions

$$\mathcal{U}^K_{h,\mathcal{Y}}(x'y) = \sum_{k=1}^{K} U_k(x')V_k(y)\,, \quad U_k \in \mathbb{V}_h\,,\ V_k \in \mathcal{S}_\mathcal{Y}$$

Our problem reads:

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha \left[\nabla_{x',y} \sum_{k=1}^{K} U_k(x')V_k(y)\right] \nabla_{x',y}\phi(x',y)\,dx'dy = d_s \int_\Omega f(x')\phi(x',0)dx'\,,$$

$\forall\, \phi \in \mathbb{V}_{h,\mathcal{Y}}$

Apply diagonalization technique: find $(v,\mu) \in \mathcal{S}_\mathcal{Y} \setminus \{0\} \times \mathbb{R}$ such that

$$\int_0^\mathcal{Y} y^\alpha v'(y)w'(y)\,dy = \mu \int_0^\mathcal{Y} y^\alpha v(y)w(y)\,dy\,, \quad \forall\, w \in \mathcal{S}_\mathcal{Y}$$

Eigenfunctions normalized so that

$$\int_0^\mathcal{Y} y^\alpha v_i'(y)v_j'(y)\,dy = \mu_j\delta_{ij}\,, \quad \int_0^\mathcal{Y} y^\alpha v_i(y)v_j(y)\,dy = \delta_{ij}$$

Diagonalization

- Write $\mathcal{U}_{h,\mathcal{Y}}^{K}(x', y) = \sum_{k=1}^{K} U_k(x')\, v_k(y)$
- Consider $\phi(x', y) = W(x')v_j(y)$, $W \in \mathbb{V}_h$
- Diagonalization decouples problem into K independent problems: For $k = 1, 2, \ldots, K$, find $U_k \in \mathbb{V}_h$ s.t.

$$a_{\mu_k,\Omega}(U_k, W) = d_s v_k(0)\langle f, W\rangle \quad , \quad \forall\, W \in \mathbb{V}_h$$

with $a_{\mu_k,\Omega}(\hat{V}, \hat{W}) = \int_\Omega \nabla_{x'} \hat{V} \cdot \nabla_{x'} \hat{W}\, dx' + \mu_k \int_\Omega \hat{V}\, \hat{W}\, dx'$

Diagonalization

- Write $\mathcal{U}_{h,\mathcal{Y}}^{K}(x',y) = \sum_{k=1}^{K} U_k(x')\, v_k(y)$
- Consider $\phi(x',y) = W(x')v_j(y)$, $W \in \mathbb{V}_h$
- Diagonalization decouples problem into K independent problems: For $k = 1, 2, \ldots, K$, find $U_k \in \mathbb{V}_h$ s.t.

$$a_{\mu_k,\Omega}(U_k, W) = d_s v_k(0)\langle f, W\rangle \quad , \quad \forall\, W \in \mathbb{V}_h$$

with $a_{\mu_k,\Omega}(\hat{V}, \hat{W}) = \int_\Omega \nabla_{x'} \hat{V} \cdot \nabla_{x'} \hat{W}\, dx' + \mu_k \int_\Omega \hat{V}\, \hat{W}\, dx'$

Why is this efficient?

Diagonalization

- Write $\mathcal{U}^K_{h,\mathcal{Y}}(x', y) = \sum_{k=1}^{K} U_k(x')\, v_k(y)$
- Consider $\phi(x', y) = W(x')v_j(y)$, $W \in \mathbb{V}_h$
- Diagonalization decouples problem into K independent problems: For $k = 1, 2, \ldots, K$, find $U_k \in \mathbb{V}_h$ s.t.

$$a_{\mu_k,\Omega}(U_k, W) = d_s v_k(0)\langle f, W\rangle \quad , \quad \forall\, W \in \mathbb{V}_h$$

with $a_{\mu_k,\Omega}(\hat{V}, \hat{W}) = \int_\Omega \nabla_{x'} \hat{V} \cdot \nabla_{x'} \hat{W}\, dx' + \mu_k \int_\Omega \hat{V}\, \hat{W}\, dx'$

Why is this efficient? Let $\hat{N} = \dim \mathbb{V}_h$ and $\hat{K} = \dim \mathcal{S}_\mathcal{Y}$.

- Trade solving an $\hat{N}\hat{K} \times \hat{N}\hat{K}$ matrix problem for $\hat{K}$ separate $\hat{N} \times \hat{N}$ matrix problems ($\hat{K} \ll \hat{N}$).

Diagonalization

- Write $\mathcal{U}_{h,\mathcal{Y}}^{K}(x', y) = \sum_{k=1}^{K} U_k(x')\, v_k(y)$
- Consider $\phi(x', y) = W(x')v_j(y)$, $W \in \mathbb{V}_h$
- Diagonalization decouples problem into K independent problems: For $k = 1, 2, \ldots, K$, find $U_k \in \mathbb{V}_h$ s.t.

$$a_{\mu_k,\Omega}(U_k, W) = d_s v_k(0)\langle f, W\rangle \quad , \quad \forall\, W \in \mathbb{V}_h$$

with $a_{\mu_k,\Omega}(\hat{V}, \hat{W}) = \int_\Omega \nabla_{x'}\hat{V} \cdot \nabla_{x'}\hat{W}\, dx' + \mu_k \int_\Omega \hat{V}\,\hat{W}\, dx'$

Why is this efficient? Let $\hat{N} = \dim \mathbb{V}_h$ and $\hat{K} = \dim \mathcal{S}_{\mathcal{Y}}$.

- Trade solving an $\hat{N}\hat{K} \times \hat{N}\hat{K}$ matrix problem for $\hat{K}$ separate $\hat{N} \times \hat{N}$ matrix problems ($\hat{K} \ll \hat{N}$).
- Embarrassingly parallelizable.

Full Discretization

Take your favorite $\mathbb{V}_h$ and let $\mathcal{S}_\mathcal{Y}$ be

- (Quasi-)Uniform mesh with $\mathbb{P}_1$

$$\|\nabla(\mathcal{U} - \mathcal{U}_{h,\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} \lesssim \left(\hat{N} \cdot \hat{K}\right)^{-\frac{s-\epsilon}{d+1}} \|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

- Graded mesh with $\mathbb{P}_1$

$$\|\nabla(\mathcal{U} - \mathcal{U}_{h,\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} \lesssim \left| \log(\hat{N} \cdot \hat{K}) \right|^s \left(\hat{N} \cdot \hat{K}\right)^{-\frac{1}{d+1}} \|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

- Geometric mesh with hp-FEM

$$\|\nabla(\mathcal{U} - \mathcal{U}^K_{h,\mathcal{Y}})\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} \lesssim |\log(\hat{N})|^q (\hat{N})^{-\frac{1}{d}} \|f\|_{\mathbb{H}^{1-s}(\Omega)}$$

for some $q > 0$.

Issue

[2] Zhimin Zhang. "How many numerical eigenvalues can we trust?" In: *Journal of Scientific Computing* 65.2 (2015), pp. 455–466.

Issue

Eigenvalue problem unstable[2]

[2]Zhang, "How many numerical eigenvalues can we trust?"

Issue

Eigenvalue problem unstable[2]

Question?

[2] Zhang, "How many numerical eigenvalues can we trust?"

Issue

Eigenvalue problem unstable[2]

Question?

Do we even need to discretize in the extended direction?

[2] Zhang, "How many numerical eigenvalues can we trust?"

Exact Eigenpairs in Extended Dimension

For $s \in (0,1)$

$$\begin{cases} -(y^\alpha \psi'(y))' = \lambda y^\alpha \psi(y) & \text{in } (0,\mathcal{Y}) \\ -y^\alpha \psi'(y) \to 0 & \text{as } y \downarrow 0 \\ \psi(\mathcal{Y}) = 0 \end{cases}$$

Expanding:

$$\begin{aligned} y^\alpha \psi''(y) + \alpha y^{\alpha-1}\psi'(y) + \lambda y^\alpha \psi(y) &= 0 \\ \psi''(y) + \frac{\alpha}{y}\psi'(y) + \lambda \psi(y) &= 0 \end{aligned}$$

Two cases: $s = 1/2$ and $s \neq 1/2$

- $s = 1/2$,

$$\mu_k = \left(\frac{(k-1/2)\pi}{\mathcal{Y}}\right)^2 \ , \ v_k(y) = \sqrt{\frac{2}{\mathcal{Y}}}\cos\left(\frac{(k-1/2)\pi}{\mathcal{Y}}\, y\right) \ , \ k \in \mathbb{N}$$

- $s \neq 1/2$,

$$\mu_k = (\eta_k/\mathcal{Y})^2 \ , \ \text{where } \eta_k \text{ is the } k^{\text{th}} \text{ root of } J_{-s}$$

$$v_k(y) = \left(\frac{\sqrt{2}}{\mu_k^{s/2}\mathcal{Y} J_{1-s}(\eta_k)}\right)(y\sqrt{\mu_k})^s J_{-s}(y\sqrt{\mu_k})$$

- $k = 1, 2, \dots, K$, solve $a_{\mu_k,\Omega}(U_k, W) = d_s v_k(0)\langle f, W\rangle \ , \ \forall\, W \in \mathbb{V}_h$.

Then, $u_{h,\mathcal{Y}}^K = \sum_{k=1}^{K} v_k(0) U_k$

Relation to a Quadrature Scheme

- Formally, write $U_k = d_s v_k(0)(\mu_k I - \Delta_h)^{-1} f_h$.

[3] Alessandra Lunardi. *Interpolation theory*. Vol. 16. Springer, 2018.

Relation to a Quadrature Scheme

- Formally, write $U_k = d_s v_k(0)(\mu_k I - \Delta_h)^{-1} f_h$.
- $u_{h,\mathcal{Y}}^K(\cdot) = \mathcal{U}_{h,\mathcal{Y}}^K(\cdot, 0) = d_s \sum_{k=1}^{K} |v_k(0)|^2 (\mu_k I - \Delta_h)^{-1} f_h$

[3] Lunardi, *Interpolation theory*.

Relation to a Quadrature Scheme

- Formally, write $U_k = d_s v_k(0)(\mu_k I - \Delta_h)^{-1} f_h$.
- $u_{h,\mathcal{Y}}^K(\cdot) = \mathcal{U}_{h,\mathcal{Y}}^K(\cdot, 0) = d_s \sum_{k=1}^{K} |v_k(0)|^2 (\mu_k I - \Delta_h)^{-1} f_h$
- Recall the Dunford-Taylor integral formula[3]

$$(-\Delta)^{-s} = \frac{2\sin(\pi s)}{\pi} \int_0^\infty t^\alpha (t^2 I - \Delta)^{-1}\, dt$$

[3] Lunardi, *Interpolation theory*.

Relation to a Quadrature Scheme

- Formally, write $U_k = d_s v_k(0)(\mu_k I - \Delta_h)^{-1} f_h$.
- $u_{h,\mathcal{Y}}^K(\cdot) = \mathcal{U}_{h,\mathcal{Y}}^K(\cdot, 0) = d_s \sum_{k=1}^{K} |v_k(0)|^2 (\mu_k I - \Delta_h)^{-1} f_h$
- Recall the Dunford-Taylor integral formula[3]

$$(-\Delta)^{-s} = \frac{2\sin(\pi s)}{\pi} \int_0^\infty t^\alpha (t^2 I - \Delta)^{-1}\, dt$$

- Interpret above series as a quadrature for

$$\int_0^\infty t^\alpha F(t) dt, \qquad F(t) = \frac{2\sin(\pi s)}{\pi} (t^2 I - \Delta)^{-1}$$

with weight t^α. The nodes and weights are:

$$t_k = \sqrt{\mu_k} \quad , \quad \omega_k = \frac{d_s |v_k(0)|^2 \pi}{2\sin(s\pi)}$$

[3] Lunardi, *Interpolation theory*.

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}} \right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}} \right)$$

where $F(x) = \frac{1}{x^2+\lambda}$.

[4] Hidenori Ogata. "A numerical integration formula based on the Bessel functions". In: *Publications of the Research Institute for Mathematical Sciences* 41.4 (2005), pp. 949–970.

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}} \right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}} \right)$$

  where $F(x) = \frac{1}{x^2+\lambda}$.

- Define[4] $\mathcal{B}_{s,d}$ as the set of functions F s.t.:

[4] Ogata, "A numerical integration formula based on the Bessel functions".

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}} \right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}} \right)$$

  where $F(x) = \frac{1}{x^2+\lambda}$.

- Define[4] $\mathcal{B}_{s,d}$ as the set of functions F s.t.:
  - $F(z)$ analytic in D_d

[4] Ogata, "A numerical integration formula based on the Bessel functions".

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}} \right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}} \right)$$

  where $F(x) = \frac{1}{x^2+\lambda}$.

- Define[4] $\mathcal{B}_{s,d}$ as the set of functions F s.t.:
  - $F(z)$ analytic in D_d
  - $0 < c < d$

[4] Ogata, "A numerical integration formula based on the Bessel functions".

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}}\right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}}\right)$$

  where $F(x) = \frac{1}{x^2+\lambda}$.

- Define[4] $\mathcal{B}_{s,d}$ as the set of functions F s.t.:
  - $F(z)$ analytic in D_d
  - $0 < c < d$
    1. $\mathcal{N}_{s,c} \equiv \int_{-\infty}^{\infty} \left(|x+ic|^{1-2s}|F(x+ic)| + |x-ic|^{1-2s}|F(x-ic)|\right) dx$

[4] Ogata, "A numerical integration formula based on the Bessel functions".

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}} \right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}} \right)$$

  where $F(x) = \frac{1}{x^2+\lambda}$.

- Define[4] $\mathcal{B}_{s,d}$ as the set of functions F s.t.:
  - $F(z)$ analytic in D_d
  - $0 < c < d$
    1. $\mathcal{N}_{s,c} \equiv \int_{-\infty}^{\infty} \left(|x+ic|^{1-2s} |F(x+ic)| + |x-ic|^{1-2s} |F(x-ic)| \right) dx$
    2. $\lim_{c \uparrow d} \mathcal{N}_{s,c} < \infty$

[4] Ogata, "A numerical integration formula based on the Bessel functions".

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}} \right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}} \right)$$

  where $F(x) = \frac{1}{x^2+\lambda}$.

- Define[4] $\mathcal{B}_{s,d}$ as the set of functions F s.t.:
  - $F(z)$ analytic in D_d
  - $0 < c < d$
    1. $\mathcal{N}_{s,c} \equiv \int_{-\infty}^{\infty} \left(|x+ic|^{1-2s} |F(x+ic)| + |x-ic|^{1-2s} |F(x-ic)| \right) dx$
    2. $\lim_{c \uparrow d} \mathcal{N}_{s,c} < \infty$
    3. $\lim_{x \to \pm\infty} \int_{-c}^{c} |x+iy|^{1-2s} |F(x+iy)| dy = 0$

[4] Ogata, "A numerical integration formula based on the Bessel functions".

Trapezoidal Quadrature

- Define

$$I_s(F) = \int_{-\infty}^{\infty} |x|^{1-2s} F(x)\, dx \ , \ \ I_{s,h}(F) = \frac{\pi}{\mathcal{Y}} \sum_{k \neq 0} \frac{2}{\mathcal{Y} \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}} \right)^{1-2s} F\left(\frac{\eta_k}{\mathcal{Y}} \right)$$

  where $F(x) = \frac{1}{x^2+\lambda}$.

- Define[4] $\mathcal{B}_{s,d}$ as the set of functions F s.t.:
  - $F(z)$ analytic in D_d
  - $0 < c < d$
    1. $\mathcal{N}_{s,c} \equiv \int_{-\infty}^{\infty} \left(|x+ic|^{1-2s} |F(x+ic)| + |x-ic|^{1-2s} |F(x-ic)| \right) dx$
    2. $\lim_{c \uparrow d} \mathcal{N}_{s,c} < \infty$
    3. $\lim_{x \to \pm\infty} \int_{-c}^{c} |x+iy|^{1-2s} |F(x+iy)| dy = 0$

- If $F \in \mathcal{B}_{s,\sqrt{\lambda_1}/2}$, then

$$|I_s(F) - I_{s,h}(F)| \lesssim \ \exp\ \left(-\mathcal{Y}\sqrt{\lambda_1} \right)$$

[4] Ogata, "A numerical integration formula based on the Bessel functions".

Error Analysis

$$\left\|u - u^K_{h,\mathcal{Y}}\right\|_{L^2(\Omega)} \leq \left\|u - u_h\right\|_{L^2(\Omega)} + \left\|u_h - u^\infty_{h,\mathcal{Y}}\right\|_{L^2(\Omega)} + \left\|u^\infty_{h,\mathcal{Y}} - u^K_{h,\mathcal{Y}}\right\|_{L^2(\Omega)}$$

[5] Matsuki and Ushijima, "A note on the fractional powers of operators approximating a positive definite selfadjoint operater".

Error Analysis

$$
\begin{aligned}
\left\| u - u_{h,\mathcal{Y}}^{K} \right\|_{L^2(\Omega)} \leq \quad & \left\| u - u_h \right\|_{L^2(\Omega)} && \sim h^{2s} \quad [5] \\
& + \left\| u_h - u_{h,\mathcal{Y}}^{\infty} \right\|_{L^2(\Omega)} && \sim \exp(-\mathcal{Y}\sqrt{\lambda_1}) \\
& + \left\| u_{h,\mathcal{Y}}^{\infty} - u_{h,\mathcal{Y}}^{K} \right\|_{L^2(\Omega)} && \sim \left(\frac{\mathcal{Y}}{K}\right)^{2s}
\end{aligned}
$$

[5] Matsuki and Ushijima, "A note on the fractional powers of operators approximating a positive definite selfadjoint operater".

Error Analysis

$$
\begin{aligned}
\left\|u - u^K_{h,\mathcal{Y}}\right\|_{L^2(\Omega)} \le \quad & \left\|u - u_h\right\|_{L^2(\Omega)} && \sim h^{2s} \quad [5] \\
& + \left\|u_h - u^\infty_{h,\mathcal{Y}}\right\|_{L^2(\Omega)} && \sim \exp(-\mathcal{Y}\sqrt{\lambda_1}) \\
& + \left\|u^\infty_{h,\mathcal{Y}} - u^K_{h,\mathcal{Y}}\right\|_{L^2(\Omega)} && \sim \left(\frac{\mathcal{Y}}{K}\right)^{2s}
\end{aligned}
$$

This suggests that we take

$$
\mathcal{Y} = 2\,s\,|\log(h)| \quad , \quad K = 2\,s\,h^{-1}\,|\log(h)|
$$

[5] Matsuki and Ushijima, "A note on the fractional powers of operators approximating a positive definite selfadjoint operater".

Square Domain

Let $\Omega = (0,1)^2$.

$$\phi_{i,j}(x_1,x_2) = \sin(i\pi x_1)\sin(j\pi x_2)\ ,\ \ \lambda_{i,j} = \pi^2(i^2+j^2)\ ,\ \ i,j\in\mathbb{N}$$

For any $s \in (0,1)$,

$$f(x_1,x_2) = (2\pi^2)^s \sin(\pi x_1)\sin(\pi x_2) \implies u(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2)$$

$\#\mathcal{T}_\Omega$	$\mathcal{Y}$	K	$L^2(\Omega)$ error	rate
16	0.69315	2	2.67052e-01	–
64	1.03972	8	1.91820e-01	0.64
256	1.38629	22	1.34186e-01	0.52
1024	1.73287	55	9.49761e-02	0.50
4096	2.07944	133	6.69246e-02	0.51
16384	2.42602	310	4.73530e-02	0.50
65536	2.77259	709	3.34751e-02	0.50

Table: $s = 0.25$

$\mathcal{Y}$	K	$L^2(\Omega)$ error	rate
2.07944	8	5.13106e-02	–
3.11916	24	1.65452e-02	1.63
4.15888	66	5.19302e-03	1.67
5.19860	166	1.70008e-03	1.61
6.23832	399	5.68776e-04	1.58
7.27805	931	1.93277e-04	1.56
8.31777	2129	6.63366e-05	1.54

Table: $s = 0.75$

Conclusions

- Recapped the spectral fractional Laplacian and the extension technique of Caffarelli-Silvestre.
- Finite element discretization and diagonalization of extended dimension.
- New semi-analytic spectral diagonalization method.
- Relation to a quadrature scheme.

Thank you for your attention!