A Semi-Analytic Diagonalization Tensor Finite Element Method for the Spectral Fractional Laplacian.

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Dissertation Defense

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Presentation Outline

Introduction and Motivation

Definition (The Fractional Laplacian)

Given $0 < s < 1$, an open, bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$, $f : \Omega \to \mathbb{R}$, find $u : \Omega \to \mathbb{R}$ such that

$$(-\Delta)^s u = f \qquad \text{in } \Omega\,, \tag{1}$$

with appropriate boundary conditions.

- Square root of Laplacian well studied.
- Relevant to applications like:
  - Image processing and denoising [19],
  - Cardiac electrophysiology [15, 16],
  - Electroconvection [11],
  - Surface quasi-geostrophic flow [8],
  - and anomalous diffusion [27].
- Many ways to interpret the fractional Laplacian [24].
- We consider the spectral definition of the fractional Laplacian.

- Recall: $-\Delta : \mathcal{D}(-\Delta) \subset L^2(\Omega) \to L^2(\Omega)$.
- Unbounded, positive, and closed operator.
- Has dense domain, and with convexity: $\mathcal{D} = H_0^1(\Omega) \cap H^2(\Omega)$.
- Has compact inverse with countable eigenpairs $\{\lambda_k, \phi_k\}_{k\geq 1}$.
- $\{\lambda_k, \phi_k\}_{k\geq 1}$ are orthonormal basis of $L^2(\Omega)$ and orthogonal basis of $H_0^1(\Omega)$ [18, Section 6.5, Theorem 1].
- $w \in L^2(\Omega) \implies w(x) = \sum_{k\geq 1} w_k(x)\phi_k(x)$, $w_k = \int_\Omega w(x)\phi_k(x)\,dx$.
- Applying $-\Delta$:

$$-\Delta w(x) = \sum_{k\geq 1} w_k(-\Delta)\phi_k(x) = \sum_{k\geq 1} \lambda_k w_k \phi_k(x)\,.$$

- Define for $w \in C_0^\infty(\Omega)$ and $s \in \mathbb{R}$,

$$(-\Delta)^s w(x) = \sum_{k \geq 1} \lambda_k^s w_k \phi_k(x) \tag{2}$$

- By density [28], define $\mathbb{H}^s(\Omega)$

$$\mathbb{H}^s(\Omega) = \left\{ w(x) = \sum_{k \geq 1} w_k \phi_k(x) \, : \, \sum_{k \geq 1} \lambda_k^s |w_k|^2 < \infty \right\} = \begin{cases} H^s(\Omega)\,, & s \in (0, \frac{1}{2})\,, \\ H_{00}^{1/2}(\Omega)\,, & s = \frac{1}{2}\,, \\ H_0^s(\Omega)\,, & s \in (\frac{1}{2}, 1)\,. \end{cases} \tag{3}$$

- $\mathbb{H}^r(\Omega)$ is dual of $\mathbb{H}^{-r}(\Omega)$, when $r < 0$.
- Given $s \in (0,1)$, if $f = \sum_{k \geq 1} f_k \phi_k \in \mathbb{H}^{-s}(\Omega)$, solution to (1) is

$$u = \sum_{k \geq 1} \lambda_k^{-s} f_k \phi_k \in \mathbb{H}^s(\Omega)\,, \tag{4}$$

with $\|u\|_{\mathbb{H}^s(\Omega)} = \|f\|_{\mathbb{H}^{-s}(\Omega)}$.

Definition (y^γ weighted L^2)

Given $D = \Omega \times \mathbb{R}^+$ and $\gamma \in \mathbb{R}$, the y^γ weighted $L^2(D)$ norm is

$$\|w\|^2_{L^2(y^\gamma,D)} = \int_D y^\gamma |w|^2 \, dx$$

Definition (Weighted Sobolev Space)

Given $D = \Omega \times \mathbb{R}^+$ and $\gamma \in \mathbb{R}$, the y^γ weighted $H^1(D)$ space is

$$H^1(y^\gamma, D) = \left\{ w \in L^2(y^\gamma, D) : \nabla w \in L^2(y^\gamma, D) \right\},$$

with norm $\|w\|_{H^1(y^\gamma,D)} = \left(\|w\|^2_{L^2(y^\gamma,D)} + \|\nabla w\|^2_{L^2(y^\gamma,D)} \right)^{1/2}$.

- **Remark**: gradient is understood in distributional sense.
- Does the space make sense?

Weighted Sobolev Spaces and Muckenhoupt Weights

- $\gamma > 0 \implies y^\gamma$ is degenerate.
- $\gamma < 0 \implies y^\gamma$ is singular.
- $\gamma \in (-1, 1) \implies y^\gamma$ is Muckenhoupt class A_2.
- Using $d\mu = y^\gamma dx$, $v \in L^2(D, \mu) \implies v \in L^1_{\text{loc}}(D)$ [20].
- $H^1(y^\gamma, D)$ is complete [23].

Appropriate Function Space

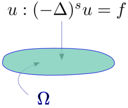
- Define $\mathcal{C} = \Omega \times (0, \infty)$.
- Lateral boundary: $\partial_L \mathcal{C} = \partial\Omega \times [0, \infty)$.
- Define: $\mathring{H}^1_L(y^\alpha, \mathcal{C}) = \{w \in H^1(y^\alpha, \mathcal{C}) : w = 0 \text{ on } \partial_L \mathcal{C}\}$.
- $\mathring{H}^1_L(y^\alpha, \mathcal{C})$ semi-norm is an equivalent norm [5, (2.21)].
- Well defined traces exist [28, Proposition 2.5]:
  tr : $\mathring{H}^1_L(y^\alpha, \mathcal{C}) \to \mathbb{H}^s(\Omega)$ such that $\forall$ smooth $w \in \mathring{H}^1_L(y^\alpha, \mathcal{C})$

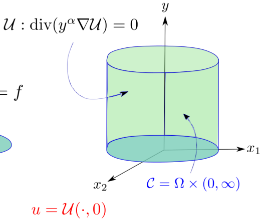
$$\text{tr } w(\mathbf{x}) = w(x, 0) \ , \ \forall \, x \in \Omega \, .$$

Caffarelli-Silvestre Extension

- How do we solve the spectral fractional Laplacian?
- Caffarelli and Silvestre [13] extended the problem to the $\mathbb{R}^{d+1}_{+}$ via a Dirichlet-to-Neumann map.
- Adapted to bounded domains in [12, 32].
- Problem now reads

$$\begin{cases} \text{div}\,(y^\alpha \nabla \mathcal{U}) = 0\,, & \text{in } \mathcal{C}\,, \\ \mathcal{U} = 0\,, & \text{on } \partial_L \mathcal{C}\,, \\ \frac{\partial \mathcal{U}}{\partial \nu^\alpha} = d_s f\,, & \text{on } \Omega \times \{0\} \end{cases}$$





where $\alpha = 1 - 2s$ and $d_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$.

- Payoff is $u(\cdot) = \mathcal{U}(\cdot, 0)$.

- Pseudo-parabolic problem transformation [34]:

$$\big(t(A-\delta\mathrm{I})+\delta\mathrm{I}\big)\frac{dw}{dt}+s(A-\delta\mathrm{I})w=0\,,$$

  where $A=-\Delta$, $\delta>0$, $w(0)=\delta^{-s}f$.
  Then, $u=w(1)$.

- (Best Uniform) Rational Approximations [2, 22, 21]
  Use finite difference method to get $h^{-2}L$.

$$-(-\Delta)^{-s}\approx-\frac{1}{h^{2s}}L^{s}\approx-\frac{1}{h^{2s}}M^{-1}K\,,$$

  where M and K found by rational approximations.

- Balakrishnan integral formulation of the operator [4]

$$(-\Delta)^{-s} = \frac{\sin(s\pi)}{\pi} \int_0^\infty t^{-s}(t\mathrm{I} - \Delta)^{-1}\, dt\,.$$

- Approximate using a quadrature formula [9, 10]
- $(t_i\mathrm{I} - \Delta)^{-1} f$ computed by FEM
- For convex domains,

$$\|u - u_h^k\|_{L^2(\Omega)} \lesssim h^2 \|f\|_{\mathbb{H}^{2-2s}(\Omega)}$$

Other Extension Based Approaches

- Hybrid FE-spectral method [3] that uses first M eigenvalues of the Laplacian.
  Approximate by FEM for lower part, Weyl's asymptotic for upper part.
  Error estimate

$$\|\nabla(\mathcal{U} - \mathcal{U}_h^M)\|_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} \lesssim |\log(M \cdot \text{dim } \mathbb{V}_h)|^q (M \cdot \text{dim } \mathbb{V}_h)^{-\min\{k, r+s\}/d} \|f\|_{\mathbb{H}^r(\Omega)}.$$

- Use generalized Laguerre polynomials [14] since weight y^α appears naturally.

- Error estimates from previous related work [28, 5]
- Piecewise linear functions on quasi-uniform intervals:

$$\|\nabla(\mathcal{U}-\mathcal{U}_h)\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim h^{s-\epsilon}\|f\|_{\mathbb{H}^{1-s}(\Omega)} \sim \left(\mathrm{dim}\mathbb{V}_h \cdot \mathrm{dim}\mathcal{S}_\mathcal{Y}\right)^{-\frac{s-\epsilon}{d+1}}\|f\|_{\mathbb{H}^{1-s}(\Omega)} .$$

- Piecewise linear functions on intervals graded towards $y=0$:

$$\|\nabla(\mathcal{U}-\mathcal{U}_h)\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim |\log(\mathrm{dim}\mathbb{V}_h \cdot \mathrm{dim}\mathcal{S}_\mathcal{Y})|^s\left(\mathrm{dim}\mathbb{V}_h \cdot \mathrm{dim}\mathcal{S}_\mathcal{Y}\right)^{-1/(d+1)}\|f\|_{\mathbb{H}^{1-s}(\Omega)} .$$

- *hp*-FEM with intervals geometrically grading towards $y=0$:

$$\|\nabla(\mathcal{U}-\mathcal{U}_h)\|_{L^2(y^\alpha,\mathcal{C}_\mathcal{Y})} \lesssim |\log(\mathrm{dim}\mathbb{V}_h)|^q\left(\mathrm{dim}\mathbb{V}_h\right)^{-1/d}\|f\|_{\mathbb{H}^{1-s}(\Omega)} ,$$

- Introduce bilinear form $a_{\mathcal{C}} : \mathring{H}^1_L(y^\alpha, \mathcal{C}) \times \mathring{H}^1_L(y^\alpha, \mathcal{C}) \to \mathbb{R}$

$$a_{\mathcal{C}}(v, w) = \int_{\mathcal{C}} y^\alpha \nabla v \cdot \nabla w \, d\mathbf{x}$$

- Find $\mathcal{U} \in \mathring{H}^1_L(y^\alpha, \mathcal{C})$ such that

$$a_{\mathcal{C}}(\mathcal{U}, v) = d_s \langle f, \text{tr } v \rangle \, , \forall v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}) \tag{5}$$

Theorem (Caffarelli-Silvestre)

Let $f \in \mathbb{H}^{-s}(\Omega)$ and assume $\mathcal{U} \in \mathring{H}^1_L(y^\alpha, \mathcal{C})$ solves (5). Then, $u = \text{tr}\, \mathcal{U} \in \mathbb{H}^s(\Omega)$ solves (1) in the sense that it satisfies (4).

Truncation

- Must truncate cylinder to finite height $\mathcal{Y} > 0$.
- Denote truncated cylinder by $\mathcal{C}_{\mathcal{Y}} = \Omega \times (0, \mathcal{Y})$.
- Define weighted Sobolev space on truncated cylinder

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}}) = \left\{ w \in H^1(y^\alpha, \mathcal{C}_{\mathcal{Y}}) : w|_{\partial_L \mathcal{C}_{\mathcal{Y}} \cup (\Omega \times \{\mathcal{Y}\})} = 0 \right\}.$$

- Problem is now: find $\mathcal{U}_{\mathcal{Y}} \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}})$ such that

$$a_{\mathcal{C}_{\mathcal{Y}}}(\mathcal{U}_{\mathcal{Y}}, v) = d_s \langle f, \text{tr } v \rangle \ , \forall v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}}) . \tag{6}$$

Proposition (Truncation)

Let $\mathcal{Y} \geq 1$. *If* $\mathcal{U} \in \mathring{H}_L^1(y^\alpha, \mathcal{C})$ *solves* (5), *then we have that*

$$\int_{\mathcal{Y}}^{\infty} \int_{\Omega} y^\alpha |\nabla \mathcal{U}|^2 \, d\mathbf{x} \lesssim e^{-\sqrt{\lambda_1}\,\mathcal{Y}} \|f\|_{\mathbb{H}^{-s}(\Omega)} .$$

Consequently, if $\mathcal{U}_\mathcal{Y} \in \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y})$ *solves* (6), *we have that*

$$\|u - \mathit{tr}\, \mathcal{U}_\mathcal{Y}\|_{\mathbb{H}^s(\Omega)} = \|\nabla(\mathcal{U} - \mathcal{U}_\mathcal{Y})\|_{L^2(y^\alpha, \mathcal{C})} \lesssim e^{-\sqrt{\lambda_1}\,\mathcal{Y}/4} \|f\|_{\mathbb{H}^{-s}(\Omega)} .$$

Proof.

See [28, Theorem 3.5]. □

Tensorial Discretization

Use the tensor product decomposition:

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}) = H^1_0(\Omega) \otimes H^1_\mathcal{Y}(y^\alpha, (0, \mathcal{Y}))\,.$$

Theorem (Tensor Products)

Let $(M_i, \mu_i), i = 1, 2,$ be measure spaces. Then, we have the isomorphism:

$$L^2(M_1, \mu_1) \otimes L^2(M_2, \mu_2) \approx L^2(M_1 \times M_2, \mu_1 \otimes \mu_2)\,,$$

provided all spaces are separable. In addition, if $\{\phi^{(i)}_k\}_{k\geq 1}$ is an orthonormal basis of $L^2(M_i, \mu_i)$, then $\{\phi^{(1)}_k \phi^{(2)}_m\}_{k,m\geq 1}$ is an orthonormal basis of their tensor product.

Tensorial Discretization

Corollary

Let

$$H^1_{\mathcal{Y}}(y^\alpha, (0, \mathcal{Y})) = \{w \in L^2(y^\alpha, (0, \mathcal{Y})) : w' \in L^2(y^\alpha, (0, \mathcal{Y})), \\ w(\mathcal{Y}) = 0\}.$$

Then, we have that

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}_{\mathcal{Y}}) = H^1_0(\Omega) \otimes H^1_{\mathcal{Y}}(y^\alpha, (0, \mathcal{Y})).$$

Remark: Proof follows analogously to above theorem - see [31, Theorem II.10] and [31, Proposition II.4.2].

Tensorial Discretization

- Approximate $H_0^1(\Omega)$ by piecewise linear elements denoting the space by $\mathbb{V}_h$.
- With quasi-uniform triangulation and $h > 0$, we have dim $\mathbb{V}_h \sim h^{-d} \sim \mathcal{N} \in \mathbb{N}$.
- In the extended dimension, take $\mathcal{S}_\mathcal{Y} \subset \mathring{H}_\mathcal{Y}^1(y^\alpha, (0, \mathcal{Y}))$ such that dim $\mathcal{S}_\mathcal{Y} \sim \mathcal{K} \in \mathbb{N}$.
- By the corollary

$$\mathbb{V}_h \otimes \mathcal{S}_\mathcal{Y} \subset H_0^1(\Omega) \otimes H_\mathcal{Y}^1(y^\alpha, (0, \mathcal{Y})) = \mathring{H}_L^1(y^\alpha, \mathcal{C}_\mathcal{Y}) \,.$$

Background

- Combining tensorial discretization with Galerkin solution yields

$$\mathcal{U}_{h,\mathcal{Y}}^{\mathcal{K}}(x,y) = \sum_{k=1}^{\mathcal{K}} U_k(x)\, v_k(y)\ ,\ u_{h,\mathcal{Y}}^{\mathcal{K}}(x) = \sum_{k=1}^{\mathcal{K}} U_k(x)\, v_k(0)\,.$$

- Discrete problem is large - $\mathcal{N}\mathcal{K} \times \mathcal{N}\mathcal{K}$.
- Introduce auxiliary eigenvalue problem: find $(v,\mu) \in \mathcal{S}_{\mathcal{Y}} \setminus \{0\} \times \mathbb{R}$ such that

$$\int_0^{\mathcal{Y}} y^\alpha v'(y) w'(y)\, dy = \mu \int_0^{\mathcal{Y}} y^\alpha v(y) w(y)\, dy\,,\ \forall w \in \mathcal{S}_{\mathcal{Y}}\,,$$
$$\int_0^{\mathcal{Y}} y^\alpha v_j'(y) v_k'(y)\, dy = \mu_k \delta_{jk}\,.$$

What does this accomplish?

$$\begin{aligned}
\int_\Omega \int_0^{\mathcal{Y}} & y^\alpha \nabla \mathcal{U}_{h,\mathcal{Y}}^{\mathcal{K}}(x,y) \cdot \nabla (V(x) v_i(y))\, dx\, dy \\
&= \int_\Omega \int_0^{\mathcal{Y}} y^\alpha \nabla \left(\sum_{k=1}^{\mathcal{K}} U_k(x) v_k(y) \right) \cdot \nabla (V(x) v_i(y))\, dx\, dy \\
&= \dots \\
&= \sum_{k=1}^{\mathcal{K}} \Bigg[\int_\Omega \nabla U_k(x) \cdot \nabla V(x) \int_0^{\mathcal{Y}} y^\alpha v_k(y) v_i(y)\, dy\, dx \\
&\qquad + U_k(x) V(x) \int_0^{\mathcal{Y}} y^\alpha v_k'(y) v_i'(y)\, dy\, dx \Bigg] \\
&= \int_\Omega \nabla U_k(x) \cdot V(x)\, dx + \mu_k \int_\Omega U_k(x) V(x)\, dx
\end{aligned}$$

Strategy

For $k = 1, 2, \ldots, \mathcal{K}$, find $U_k \in \mathbb{V}_h$ such that

$$\int_\Omega \nabla U_k(x) \cdot \nabla V(x)\, dx + \mu_k \int_\Omega U_k(x) V(x)\, dx = d_s v_k(0) \langle f, V \rangle\,, \quad (7)$$

$\forall V \in \mathbb{V}_h$.

Remark:

- traded problem of size $\mathcal{N}\mathcal{K} \times \mathcal{N}\mathcal{K}$ for $\mathcal{K}$ problems of size $\mathcal{N} \times \mathcal{N}$.
- each problem in k is independent, may be solved in parallel

Issue

- Generalized eigenvalue problem: $\Lambda M\tilde{V} = K\tilde{V}$, $\Lambda_{i,i} = \mu_i$.
- Graded, or geometric, meshes $\implies \kappa(M), \kappa(K) \gg 1$.
- With a uniform mesh, $\mu_i \to \infty$, as $i \to \infty$. Cannot compute many accurately [35].
- Studied in more detail with attempts to control with suitable paramters [6].

Exact Eigenpairs in Extended Dimension

For $s \in (0,1)$ and $s \neq 1/2$,

$$\begin{cases} -(y^\alpha \psi'(y))' = \mu y^\alpha \psi(y) & \text{in } (0, \mathcal{Y}) \\ -y^\alpha \psi'(y) \to 0 & \text{as } y \downarrow 0 \\ \psi(\mathcal{Y}) = 0 \end{cases}$$

Expanding:

$$\begin{aligned} y^\alpha \psi''(y) + \alpha y^{\alpha-1} \psi'(y) + \mu y^\alpha \psi(y) &= 0 \\ \psi''(y) + \frac{\alpha}{y} \psi'(y) + \mu \psi(y) &= 0 \end{aligned}$$

Eigenpairs

Theorem

There exists a countable number $\{(\mu_k, \psi_k)\}_{k \geq 1} \subset \mathbb{R}_+ \times H^1_{\mathcal{Y}}(y^\alpha, (0, \mathcal{Y}))$ *solutions to the ODE. The eigenvalues can be numbered so that*

$$0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \leq \mu_k \leq \cdots, \; \mu_k \to \infty \text{ as } k \to \infty \, .$$

Proof.

The ODE is a singular Sturm-Liouville problem with mixed boundary conditions. Follows arguments given in [33, Chapter 11]. □

Solutions

- Analytic solutions depend on the value of s.
- $s \neq \frac{1}{2}$ requires work.
- $s = \frac{1}{2}$ is straight-forward.

$$\psi''(y) + \mu\psi(y) = 0\,.$$

Lemma ($s = \frac{1}{2}$)

Let $s = \frac{1}{2}$. The eigenpairs $\{(\mu_k, \psi_k)\}_{k=1}^{\infty}$ that solve the ODE are

$$\psi_k(y) = \sqrt{\frac{2}{\mathcal{Y}}}\cos(\mu_k y) \quad \textit{and} \quad \mu_k = \left(\frac{(k-\frac{1}{2})\pi}{\mathcal{Y}}\right)^2,$$

for all $k \in \mathbb{N}$.

$$s \neq \tfrac{1}{2} \implies \psi''(y) + \frac{\alpha}{y}\psi'(y) + \mu\psi(y) = 0\,.$$

Lemma ($s \neq \frac{1}{2}$)

Let $s \neq \frac{1}{2}$. The eigenpairs $\{(\mu_k, \psi_k)\}_{k=1}^{\infty}$ that solve the ODE are

$$\psi_k(y) = \left(\frac{\sqrt{2}}{\mu_k^{s/2}\mathcal{Y} J_{1-s}(\eta_k)}\right)(y\sqrt{\mu_k})^s J_{-s}(y\sqrt{\mu_k})\ ,\ k \in \mathbb{N}\,,$$

where η_k is the k^{th} positive root of J_{-s} and the eigenvalues are given by

$$\mu_k = \left(\frac{\eta_k}{\mathcal{Y}}\right)^2\,.$$

Proof.

- Ansatz $\psi(y) = z^s w(z)$, where $z = \sqrt{\mu} y$.
- ODE becomes $z^s w''(z) + z w'(z) + (z^2 - s^2) w(z) = 0$.
- Bessel equation [1, Section 9.1], so [17, Section 10.2(ii)],

$$w(z) = A J_s(z) + B Y_s(z)\,,$$

- Satisfy the Neumann boundary condition first:

$$-\lim_{y \downarrow 0} y^\alpha \psi'(y) = \frac{2^{1-s}\mu^s \big(A\pi + B\cos(\pi s)\Gamma(1-s)\Gamma(s)\big)}{\pi\Gamma(s)} = 0\,,$$

- $\implies A\pi + B\cos(\pi s)\Gamma(1-s)\Gamma(s) = 0$.

□

continued.

- Dirichlet BC $\implies AJ_s(\sqrt{\mu}\mathcal{Y}) + BY_s(\sqrt{\mu}\mathcal{Y}) = 0.$

- Express B in terms of A

$$\psi(y) = \frac{2^s A}{\Gamma(1-s)\cos(\pi s)} {}_0F_1(0, 1-s; -\mu y^2/4)\,,$$

- Use the relation [1, Equation (9.1.69)]

$$J_{-s}(\sqrt{\mu}y) = \frac{\left(\frac{\sqrt{\mu}y}{2}\right)^{-s}}{\Gamma(1-s)} {}_0F_1(0, 1-s; -\mu y^2/4)\,.$$

- Then, $\psi(y) = \frac{A}{\cos(\pi s)} \left(\sqrt{\mu}y\right)^s J_{-s}(\sqrt{\mu}y)\,.$

- Choose μ such that $\sqrt{\mu_k}\,\mathcal{Y} = \eta_k$ and A_k so that $\|\psi_k\|^2_{L^2(y^\alpha,(0,\mathcal{Y}))} = 1.$

□

Recall, $u_{h,\mathcal{Y}}^{\mathcal{K}}(x) = \sum_{k=1}^{\mathcal{K}} U_k(x)\psi_k(0)$.

Corollary

If $\{(\mu_k, \psi_k)\}_{k=1}^{\infty}$ solve the ODE, then

$$\psi_k^{(1/2)}(0) = \sqrt{\frac{2}{\mathcal{Y}}} \quad , \quad \psi_k^{(s)}(0) = \frac{2^{s+1/2}}{\mu_k^{s/2}\mathcal{Y} J_{1-s}(\eta_k)\Gamma(1-s)} .$$

Proof.

Use the asymptotic form for Bessel functions of the first kind [1, Equation (9.1.7)].

$$J_{-s}(z) \sim \frac{1}{\Gamma(1-s)} \left(\frac{2}{z}\right)^s ,$$

and use this to take the limit $y \downarrow 0$. □

Summary of approach.

- Solving $(-\Delta)^s u = f$ in Ω.

- Use Caffarelli-Silvestre extension - find $\mathcal{U} \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$.

Algorithm 1 Pseudo-code for computation.

Input: $\Omega, s, f, h, \mathcal{Y}, \mathcal{K}$
Output: $u^{\mathcal{K}}_{h,\mathcal{Y}}$

```
1: for k = 1 to K do
2:     Compute {μ_k, ψ_k}
3:     Solve ∫_Ω ∇U_k · ∇V dx + μ_k ∫_Ω U_k V dx = d_s ψ_k(0)⟨f, V⟩
4: end for
5: Calculate u^K_{h,Y}(x) = Σ_{k=1}^{K} U_k(x) ψ_k(0)
```

1: **for** $k = 1$ to $\mathcal{K}$ **do**
2: Compute $\{\mu_k, \psi_k\}$
3: Solve $\int_\Omega \nabla U_k \cdot \nabla V \, dx + \mu_k \int_\Omega U_k V \, dx = d_s \psi_k(0) \langle f, V \rangle$
4: **end for**
5: Calculate $u^{\mathcal{K}}_{h,\mathcal{Y}}(x) = \sum_{k=1}^{\mathcal{K}} U_k(x) \psi_k(0)$

Discrete Laplacian

- $\Delta_h : \mathbb{V}_h \to \mathbb{V}_h$

$$\int_\Omega \Delta_h W V \, dx = -\int_\Omega \nabla W \cdot \nabla V \, dx \ , \ \forall \, V, W \in \mathbb{V}_h$$

- $\exists \, \{(\lambda_{h,m}, \Phi_{h,m})\}_{m=1}^{M} \subset \mathbb{R}_+ \times \mathbb{V}_h \setminus \{0\}$ such that

$$\int_\Omega \nabla \Phi_{h,m} \cdot \nabla V dx = \lambda_{h,m} \int_\Omega \Phi_{h,m} V dx \ , \ \forall V \in \mathbb{V}_h$$

- $\{\Phi_{h,m}\}_{m=1}^{M} \subset \mathbb{V}_h$ is an orthonormal basis of $\mathbb{V}_h$ in $L^2(\Omega)$
- $0 < \lambda_{h,1} \leq \lambda_{h,2} \leq \cdots \leq \lambda_{h,M}$
- $\lambda_{h,1} \geq C_P^{-1}$ and $\lambda_{h,M} \lesssim h^{-2}$, where C_P is the Poincaré constant
- $(-\Delta_h)^r$, $r \in \mathbb{R}$, given by standard spectral theory

Error Decomposition

- Let $P_h f$ be the $L^2(\Omega)-$projection of f.
- Denote the discrete solution as $u_h = (-\Delta_h)^{-s} P_h f$.
- Use $U_k = d_s \psi_k(0)(\mu_k \mathrm{I} - \Delta_h)^{-1} P_h f$ to write

$$u_{h,\mathcal{Y}}^{\mathcal{K}} = d_s \sum_{k=1}^{\mathcal{K}} |\psi_k(0)|^2 (\mu_k \mathrm{I} - \Delta_h)^{-1} P_h f$$

- Write

$$\|u - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)} \leq \|u - u_h\|_{L^2\Omega} + \|u_h - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)} + \|u_{h,\mathcal{Y}}^{\infty} - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)}$$

$\leq$ Discretization Error $+$ Quadrature Error $+$ Truncation Error

- Discretization error controlled [26] by

$$\|u - u_h\|_{L^2(\Omega)} \lesssim h^{2s} \|f\|_{L^2(\Omega)}$$

Quadrature Error

- Balakrishnan formula for positive operators [25] (after change of variables $r = t^2$):

$$(-\Delta_h)^{-s} = \frac{2\sin(\pi s)}{\pi} \int_0^\infty t^\alpha (t^2 \mathrm{I} - \Delta_h)^{-1} dt$$

- Interpret solution representation is a quadrature formula
- Notation

$$F_\lambda(t) = \frac{2\sin(\pi s)}{\pi} \frac{1}{t^2 + \lambda}$$

$$I_s(F_\lambda) = \int_0^\infty t^\alpha F_\lambda(t)\, dt \ , \ Q_s^{\mathcal{K}}(F_\lambda) = \sum_{k=1}^{\mathcal{K}} (t_k^{(s)})^\alpha \omega_k^{(s)} F_\lambda(t_k^{(s)})$$

Theorem (Error Estimate)

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex polytope and $f \in L^2(\Omega)$. Let, for $s \in (0,1)$, $u \in \mathbb{H}^s(\Omega)$ be the solution and $u^{\mathcal{K}}_{h,\mathcal{Y}} \in \mathbb{V}_h$ be as defined above. Then, we have

$$\|u - u^{\mathcal{K}}_{h,\mathcal{Y}}\|_{L^2(\Omega)} \lesssim \Bigg(h^{2s} + \sup_{\lambda \geq \frac{1}{C_P}} |I_s(F_\lambda) - Q^{\infty}_s(F_\lambda)| + \sup_{\lambda \geq \frac{1}{C_P}} \left| \sum_{k > \mathcal{K}} (t^{(s)}_k)^{1-2s} \omega^{(s)}_k F_\lambda(t^{(s)}_k) \right| \Bigg) \|f\|_{L^2(\Omega)},$$

where the implicit constant is independent of $\mathcal{Y}$, $\mathcal{K}$, h, and f.

Proof.

$$(i) \quad u_h = (-\Delta_h)^{-s} P_h f = \int_0^\infty t^\alpha F_h(t) P_h f \, dt = \sum_{m=1}^{M} f_m I_s(F_{\lambda_{h,m}}) \Phi_{h,m}$$

$$(ii) \; u_{h,\mathcal{Y}}^{\mathcal{K}} = \sum_{m=1}^{M} f_m Q_s^{\mathcal{K}}(F_{\lambda_{h,m}}) \Phi_{h,m}$$

$$\begin{aligned}(iii) \; \|u_h - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)}^2 &= \sum_{m=1}^{M} |f_m|^2 \left| I_s(F_{\lambda_{h,m}}) - Q_s^{\mathcal{K}}(F_{\lambda_{h,m}}) \right|^2 \\ &\leq \sup_{\lambda > \frac{1}{C_P}} \left| I_s(F_\lambda) - Q_s^{\mathcal{K}}(F_\lambda) \right|^2 \|f\|_{L^2(\Omega)}^2\end{aligned}$$

□

Lemma (Quadrature Error for $s = \frac{1}{2}$)

Let $s = \frac{1}{2}$ and $\lambda > 0$. With the weights

$$t_k^{(\frac{1}{2})} = \frac{(k - \frac{1}{2})\pi}{\mathcal{Y}} \quad , \quad \omega_k^{(\frac{1}{2})} = \frac{\pi}{\mathcal{Y}} \, ,$$

we have that

$$\left| I_{\frac{1}{2}}(F_\lambda) - Q^{\infty}_{\frac{1}{2}}(F_\lambda) \right| \lesssim \frac{\exp(-\sqrt{\lambda}\,\mathcal{Y})}{\sqrt{\lambda}}$$

Remark:

$$I_{\frac{1}{2}}(F_\lambda) = \frac{2}{\pi} \int_0^\infty \frac{1}{t^2 + \lambda} dt = \frac{1}{\sqrt{\lambda}}$$

Proof.

Using partial fraction expansion [30, Equation (30:6:5)] of $\tanh(z)$:

$$Q^{\infty}_{\frac{1}{2}}(F_\lambda) = \frac{\pi}{\mathcal{Y}} \sum_{k=1}^{\infty} \frac{2}{\pi} \frac{1}{\frac{(k-\frac{1}{2})^2\pi^2}{\mathcal{Y}^2} + \lambda} = \cdots = \frac{1}{\sqrt{\lambda}} \tanh(\sqrt{\lambda}\,\mathcal{Y})$$

$$\begin{aligned} \left| I_{\frac{1}{2}}(F_\lambda) - Q^{\infty}_{\frac{1}{2}}(F_\lambda) \right| &= \frac{1}{\sqrt{\lambda}}(1 - \tanh(\sqrt{\lambda}\mathcal{Y})) \\ &= \frac{1}{\sqrt{\lambda}} \left(\frac{2\exp(-2\sqrt{\lambda}\mathcal{Y})}{1 + \exp(-2\sqrt{\lambda}\mathcal{Y})} \right) \lesssim \frac{\exp(-\sqrt{\lambda}\mathcal{Y})}{\sqrt{\lambda}} \end{aligned}$$

□

Lemma (Truncation Error for $s = \frac{1}{2}$)

Let $s = \frac{1}{2}$, $\lambda > 0$, and $\mathcal{K} \in \mathbb{N}$. Using the same weights as above, and assuming $\mathcal{K}$ is sufficiently large, we have that

$$\left| Q^{\infty}_{\frac{1}{2}}(F_\lambda) - Q^{\mathcal{K}}_{\frac{1}{2}}(F_\lambda) \right| \lesssim \frac{\mathcal{Y}}{\mathcal{K}} .$$

Proof.

Using the digamma function [7, Section 1.9, formula (10)] and its asymptotic for large arguments [1, Formula 6.4.12]:

$$\begin{aligned} \left| Q^{\infty}_{\frac{1}{2}}(F_\lambda) - Q^{\mathcal{K}}_{\frac{1}{2}}(F_\lambda) \right| &= \frac{2}{\mathcal{Y}} \sum_{k > \mathcal{K}} \frac{1}{\frac{(k-\frac{1}{2})^2 \pi^2}{\mathcal{Y}^2} + \lambda} \\ &\leq \frac{2\mathcal{Y}}{\pi^2} \sum_{k \geq \mathcal{K}} \frac{1}{(k+\frac{1}{2})^2} \leq \frac{2\mathcal{Y}}{\pi^2} \psi'(\mathcal{K}) \lesssim \frac{\mathcal{Y}}{\mathcal{K}} , \end{aligned}$$

□

Corollary (error estimate for $s = \frac{1}{2}$)

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex polytope and $f \in L^2(\Omega)$. Let $u \in \mathbb{H}^{1/2}(\Omega)$ be the solution for $s = 1/2$. Let $u_{h,\mathcal{Y}}^{\mathcal{K}}$ be given by the representation given above, where $\mathbb{V}_h$ is constructed using a quasi-uniform mesh of size h. Then, we have that

$$\|u - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)} \lesssim \left(h + \sqrt{C_P} \exp\left(-\frac{\mathcal{Y}}{\sqrt{C_P}} \right) + \frac{\mathcal{Y}}{\mathcal{K}} \right) \|f\|_{L^2(\Omega)} ,$$

where C_P is the Poincaré constant. In particular, if the truncation parameter is chosen as $\mathcal{Y} \sim |\log h|$, and the number of quadrature points is chosen as $\mathcal{K} \sim \frac{|\log h|}{h}$, we may conclude that

$$\|u - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)} \lesssim h\|f\|_{L^2(\Omega)} .$$

Error Analysis for $s \neq \frac{1}{2}$

Lemma

For $s \neq \frac{1}{2}$, the solution representation

$$u_{h,\mathcal{Y}}^{\mathcal{K}} = d_s \sum_{k=1}^{\mathcal{K}} |\psi_k(0)|^2 (\mu_k \mathrm{I} - \Delta_h)^{-1} P_h f$$

can be written as

$$u_{h,\mathcal{Y}}^{\mathcal{K}} = \sum_{m=1}^{M} f_m Q_s^{\mathcal{K}}(F_{\lambda_{h,m}}) \Phi_{h,m} \,,$$

where the nodes and weights are given by

$$t_k^{(s)} = \sqrt{\mu_k} = \frac{\eta_k}{\mathcal{Y}} \quad \text{and} \quad \omega_k^{(s)} = \frac{2}{\mathcal{Y} \eta_k J_{1-s}^2(\eta_k)} \,.$$

Proof.

Substitute in nodes and weights, apply Euler reflection [1, Formula 6.1.17] to get

$$u_{h,\mathcal{Y}}^{\mathcal{K}} = \frac{2\sin(\pi s)}{\pi} \sum_{k=1}^{\mathcal{K}} \frac{2}{\mu_k^s \mathcal{Y}^2 J_{1-s}^2(\eta_k)} (\mu_k \mathrm{I} - \Delta_h)^{-1} P_h f \, .$$

$$\begin{aligned} u_{h,\mathcal{Y}}^{\mathcal{K}} &= \sum_{k=1}^{\mathcal{K}} \left(\frac{\eta_k}{\mathcal{Y}}\right)^{1-2s} \frac{2}{\eta_k \mathcal{Y} J_{1-s}^2(\eta_k)} \left[\frac{2\sin(\pi s)}{\pi} \sum_{m=1}^{M} f_m \frac{1}{\left(\frac{\eta_k}{\mathcal{Y}}\right)^2 + \lambda_{h,m}} \right] \Phi_{h,m} \\ &= \sum_{m=1}^{M} f_m \sum_{k=1}^{\mathcal{K}} \left(\frac{\eta_k}{\mathcal{Y}}\right)^{1-2s} \frac{2}{\eta_k \mathcal{Y} J_{1-s}^2(\eta_k)} F_{\lambda_{h,m}} \left(\frac{\eta_k}{\mathcal{Y}}\right) \Phi_{h,m} \\ &= \sum_{m=1}^{M} f_m \sum_{k=1}^{\mathcal{K}} \left(t_k^{(s)}\right)^{1-2s} \omega_k^{(s)} F_{\lambda_{h,m}}(t_k^{(s)}) \Phi_{h,m} = \sum_{m=1}^{M} f_m Q_s^{\mathcal{K}}(F_{\lambda_{h,m}}) \Phi_{h,m} \end{aligned}$$

□

- Relate to quadrature formula studied in [29] for $\nu > -1$ and $\tau > 0$:

$$\int_{-\infty}^{\infty} |t|^{1+2\nu} G(t)\, dt \approx \tau \sum_{k\in\mathbb{Z}\setminus\{0\}} w_{\nu,k} |\tau\xi_{\nu,k}|^{1+2\nu} G(\tau\xi_{\nu,k})$$

- Weights

$$w_{\nu,k} = \frac{2}{\pi^2 \xi_{\nu,|k|} J_{\nu+1}^2(\pi\xi_{\nu,|k|})}\ ,\ k \in \mathbb{Z} \setminus \{0\}$$

- Nodes are, for $k \in \mathbb{Z} \setminus \{0\}$, zeros of $J_\nu(\pi x)$ so that

$$\cdots \leq \xi_{\nu,-2} \leq \xi_{\nu,-1} < 0 < \xi_{\nu,1} \leq \xi_{\nu,2} \leq \cdots$$

and $\xi_{\nu,-k} = -\xi_{\nu,k}$.

- If G is even

$$\int_0^{\infty} t^{1+2\nu} G(t)\, dt \approx \tau \sum_{k=1}^{\infty} w_{\nu,k} |\tau\xi_{\nu,k}|^{1+2\nu} G(\tau\xi_{\nu,k})$$

Lemma (Quadrature Equivalence)

Let $s \neq \frac{1}{2}$. Then with the weights and nodes defined previously, the two quadrature formulae are equivalent with $\nu = -s > -1$ and $\tau = \frac{\pi}{\mathcal{Y}}$.

Proof.

- $J_{-s}(\eta_k) = 0 \implies \eta_k = \pi \xi_{-s,k}$
- $\frac{\eta_k}{\mathcal{Y}} = \tau \xi_{-s,k}$

$$\begin{aligned}
&\tau \sum_{k=1}^{\mathcal{K}} \frac{2}{\pi^2 \xi_{-s,k} J_{1-s}^2(\pi \xi_{-s,k})} \left|\tau \xi_{-s,k}\right|^{1-2s} F_\lambda(\tau \xi_{-s,k}) \\
&\qquad = \sum_{k=1}^{\mathcal{K}} \frac{\tau}{\pi} \frac{2}{\pi \xi_{-s,k} J_{1-s}^2(\pi \xi_{-s,k})} \left|\tau \xi_{-s,k}\right|^{1-2s} F_\lambda(\tau \xi_{-s,k}) \\
&\qquad = \sum_{k=1}^{\mathcal{K}} \frac{2}{\mathcal{Y} \eta_k J_{1-s}^2(\eta_k)} \left(\frac{\eta_k}{\mathcal{Y}}\right)^{1-2s} F_\lambda\left(\frac{\eta_k}{\mathcal{Y}}\right) = Q_s^{\mathcal{K}}(F_\lambda).
\end{aligned}$$

□

Definition (class $\mathcal{B}_{s,\ell}$)

Let $\ell > 0$ and denote

$$D_\ell = \{z \in \mathbb{C} : |\Im z| < \ell\}\,, \qquad \Gamma_\ell = \partial D_\ell\,.$$

By $\mathcal{B}_{s,\ell}$ we denote the collection of functions $G : \bar{D}_\ell \to \mathbb{C}$ that satisfy:

1. G is analytic in the strip D_ℓ.
2. For all $0 < c < \ell$, the integral

$$\mathcal{N}_{s,c}(G) = \int_{-\infty}^{\infty} \left[|x + \imath c|^{1-2s}|G(x + \imath c)| + |x - \imath c|^{1-2s}|G(x - \imath c)|\right]\, dx$$

   exists. In addition, we require that $\mathcal{N}_{s,\ell-0}(G) = \lim_{c\uparrow\ell} \mathcal{N}_{s,c}(G)$ exists and remains finite.
3. For all $0 < c < \ell$, we require that

$$\lim_{x\to\pm\infty} \int_{-c}^{c} |x + \imath y|^{1-2s}|G(x + \imath y)|\, dy = 0$$

What is the utility?

Theorem (Quadrature Error)

Let $\ell > 0$*,* $s \in (0,1)$*, and* $G \in \mathcal{B}_{s,\ell}$*. Then,*

$$|I_s(G) - Q_s^\infty(G)| \lesssim \mathcal{N}_{-s,\ell-0}(G)\exp(-2\ell\mathcal{Y}),$$

where the implicit constant depends only on s *and* ℓ*.*

Proof.

See [29, Theorem 2.1]. □

Theorem ($F_\lambda \in \mathcal{B}_{s,\ell}$)

Let $s \in (0,1)$, $C_P^{-1} > 0$ and $\ell = \frac{1}{2}\sqrt{C_P^{-1}}$. For every $\lambda \geq C_P^{-1}$, we have the $F_\lambda \in \mathcal{B}_{s,\ell}$. Moreover, $\mathcal{N}_{s,\ell-0}(F_\lambda)$ depends only on s and ℓ.

Proof.

- Poles of F_λ are $\pm\imath\sqrt{\lambda}$.
- $\implies F_\lambda$ is analytic in the strip.
- Remaining requirements straightforward to show, but tedious.

□

Corollary (Quadrature Error)

Let $s \neq \frac{1}{2}$ and the nodes and weights of the quadrature be as above. Then,

$$\sup_{\lambda \geq \frac{1}{C_P}} |I_s(F_\lambda) - Q_s^\infty(F_\lambda)| \lesssim \exp\left(-\frac{\mathcal{Y}}{\sqrt{C_P}}\right)$$

Lemma (Truncation Error)

Let $s \neq \frac{1}{2}$ and $\mathcal{K} \in \mathbb{N}$. Let the nodes and weights of the quadrature scheme be as above. If $\mathcal{K}$ is sufficiently large, then

$$\sup_{\lambda \geq \frac{1}{C_P}} |Q_s^\infty(F_\lambda) - Q_s^{\mathcal{K}}(F_\lambda)| \lesssim \left(\frac{\mathcal{Y}}{\mathcal{K}}\right)^{2s}$$

Proof: Truncation Error.

- Asymptotic form of Bessel function [1]: $J_{-s}(z) \sim \sqrt{\frac{2}{\pi z}}$.

$$|Q_s^{\infty}(F_\lambda) - Q_s^{\mathcal{K}}(F_\lambda)| \leq \frac{\pi}{\mathcal{Y}} \sum_{k>\mathcal{K}} \frac{2}{\pi \eta_k J_{1-s}(\eta_k)^2} \left(\frac{\eta_k}{\mathcal{Y}}\right)^{1-2s} F_\lambda\left(\frac{\eta_k}{\mathcal{Y}}\right)$$
$$\lesssim \frac{1}{\mathcal{Y}} \sum_{k>\mathcal{K}} \left(\frac{\eta_k}{\mathcal{Y}}\right)^{1-2s} \frac{1}{\frac{\eta_k^2}{\mathcal{Y}^2} + \lambda} = \mathcal{Y}^{2s} \sum_{k>\mathcal{K}} \frac{\eta_k^{1-2s}}{\eta_k^2 + \mathcal{Y}^2\lambda}.$$

- McMahon's asymptotic [17], $\eta_k \sim \pi k$, and Hurwitz Zeta function [17], ζ:

$$\sum_{k>\mathcal{K}} \frac{\eta_k^{1-2s}}{\eta_k^2 + \mathcal{Y}^2\lambda} \lesssim \sum_{k>\mathcal{K}} \frac{1}{k^{1+2s}} = \zeta(1+2s, \mathcal{K}+1)$$

- Hurwitz Zeta asymptotic for large $\mathcal{K}$ [17]:

$$\zeta(1+2s, \mathcal{K}+1) \sim (\mathcal{K}+1)^{-2s} < \mathcal{K}^{-2s}$$

□

Corollary

Let $\Omega \subset \mathbb{R}^d$ be a bounded, convex polytope and $f \in L^2(\Omega)$. Let $u \in \mathbb{H}^s(\Omega)$ be the solution for $s \in (0,1) \setminus \{\frac{1}{2}\}$. Let $u_{h,\mathcal{Y}}^{\mathcal{K}}$ be given as above, where $\mathbb{V}_h$ is constructed using a quasi-uniform mesh of size h. Then, we have that

$$\|u - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)} \lesssim \left(h^{2s} + \exp\left(-\frac{\mathcal{Y}}{\sqrt{C_P}}\right) + \left(\frac{\mathcal{Y}}{\mathcal{K}}\right)^{2s} \right) \|f\|_{L^2(\Omega)} ,$$

where C_P is the Poincaré constant. In particular, if the truncation parameter is chosen as $\mathcal{Y} \sim 2s|\log h|$ and the number of quadrature points is chosen as $\mathcal{K} \sim \frac{\mathcal{Y}}{h}$, we may conclude that

$$\|u - u_{h,\mathcal{Y}}^{\mathcal{K}}\|_{L^2(\Omega)} \lesssim h^{2s} \|f\|_{L^2(\Omega)} .$$

Convergence Study

- Use $\Omega_L \subset \mathbb{R}^2$, defined by

$$\mathbf{c} \in \{(0,0), (1,0), (-1,1), (-1,-1), (0,-1)\}.$$

- Prescribe

$$\begin{aligned} f(x_1,x_2) = (2\pi^2)^s \sin(\pi x_1)\sin(\pi x_2) \\ + \left((3^2+2^2)\pi^2\right)^s \sin(3\pi x_1)\sin(2\pi x_1) \\ + \left((5^2+4^2)\pi^2\right)^s \sin(5\pi x_1)\sin(4\pi x_2). \end{aligned}$$

- Exact solution is

$$\begin{aligned} u(x_1,x_2) = \sin(\pi x_1)\sin(\pi x_2) + \sin(3\pi x_1)\sin(2\pi x_2) \\ + \sin(5\pi x_1)\sin(4\pi x_2). \end{aligned}$$

$\# \mathcal{T}_{\Omega_L}$	$\mathcal{Y}$	$\mathcal{K}$	Error	Rate
65	0.69315	2	1.18473e+00	–
225	1.03972	8	9.22409e-01	0.36
833	1.38629	22	6.59965e-01	0.48
3201	1.73287	55	4.68283e-01	0.50
12545	2.07944	133	3.29807e-01	0.51
49665	2.42602	310	2.33219e-01	0.50
197633	2.77259	709	1.64819e-01	0.50
788481	3.11916	1597	1.16468e-01	0.50
3149825	3.46574	3548	8.23623e-02	0.50

(a) $s = 0.25$

65	2.07944	8	8.62909e-01	–
225	3.11916	24	3.54492e-01	1.28
833	4.15888	66	1.15435e-01	1.62
3201	5.19860	166	3.69013e-02	1.65
12545	6.23832	399	1.19066e-02	1.63
49665	7.27805	931	3.91190e-03	1.61
197633	8.31777	2129	1.30633e-03	1.58
788481	9.35749	4791	4.42652e-04	1.56
3149825	10.3972	10646	1.51727e-04	1.54

(b) $s = 0.75$

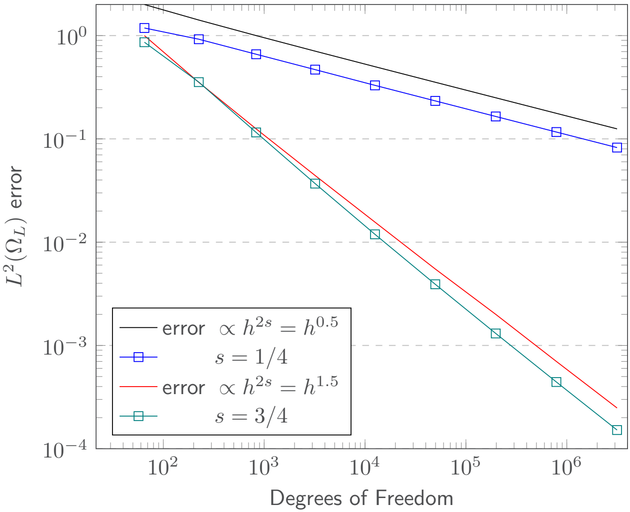


Figure: Error in the $L^2(\Omega_L)$ norm versus the number of degrees of freedom using $\mathbb{Q}_1$ finite elements for $s = 1/4$ and $s = 3/4$ on uniformly refined meshes of Ω_L.

Convergence Study

- Use $\Omega_C = \{x \in \mathbb{R}^2 : |x| < 1\}$.
- Using polar coordinates

$$\phi_{m,n}(r,\theta) = J_m(j_{m,n} r)\big(A_{m,n}\cos(m\theta) + B_{m,n}\sin(m\theta)\big)\,.$$

- J_m denotes the Bessel function of the first kind with parameter m, $j_{m,n}$ is the n^{th} zero, and $\lambda_{m,n} = (j_{m,n})^2$.
- Take $f = (\lambda_{1,1})^s \phi_{1,1}$.
- Exact solution is $u(r,\theta) = \phi_{1,1}(r,\theta)$.

# $\mathcal{T}_{\Omega_C}$	$\mathcal{Y}$	$\mathcal{K}$	Error	Rate
89	0.69315	2	3.95613e-01	–
337	1.03972	8	2.49627e-01	0.66
1313	1.38629	22	1.74086e-01	0.52
5185	1.73287	55	1.23124e-01	0.50
20609	2.07944	133	8.67400e-02	0.51
82177	2.42602	310	6.13695e-02	0.50
328193	2.77259	709	4.33828e-02	0.50
1311745	3.11916	1597	3.06600e-02	0.50

(c) $s = 0.25$

# $\mathcal{T}_{\Omega_C}$	$\mathcal{Y}$	$\mathcal{K}$	Error	Rate
89	2.07944	8	6.73748e-02	–
337	3.11916	24	2.09143e-02	1.69
1313	4.15888	66	6.38904e-03	1.71
5185	5.19860	166	2.04193e-03	1.65
20609	6.23832	399	6.70252e-04	1.61
82177	7.27805	931	2.24480e-04	1.58
328193	8.31777	2129	7.62235e-05	1.56
1311745	9.35749	4791	2.61759e-05	1.54

(d) $s = 0.75$

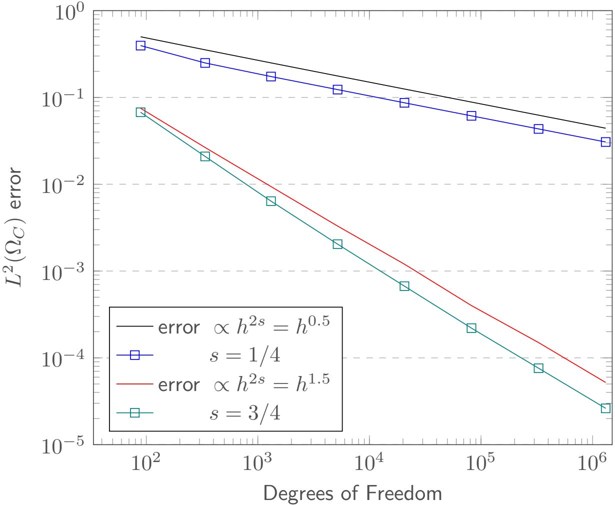


Figure: Error in the $L^2(\Omega_C)$ norm versus the number of degrees of freedom using $\mathbb{Q}_1$ finite elements for $s = 1/4$ and $s = 3/4$ on uniformly refined meshes of Ω_C.

Preliminary 3-D Results

- $\Omega = (0,1)^3$, $s = 0.75$.
- $f(x_1, x_2) = (2\pi^2)^s(\sin(\pi x_1)\sin(\pi x_2))$,
- $u(x_1, x_2) = \sin(\pi x_1)\sin(\pi x_2)$.

$\# \mathcal{T}_{\Omega_C}$	$\mathcal{Y}$	$\mathcal{K}$	Error	Rate
125	2.07944	8	4.36048e-02	–
729	3.11916	24	1.45719e-02	1.58
4913	4.15888	66	4.66818e-03	1.64
35937	5.1986	166	1.55315e-03	1.59
274625	6.23832	399	5.26152e-04	1.56
2146689	7.27805	931	1.80505e-04	1.54

Remark: each MPI rank requires ~ 3 GB RAM for finest grid.

- Use $\Omega_{\text{square}} = (0,1)^2 \subset \mathbb{R}^2$.
- Recall

$$\phi_{m,n}(x_1, x_2) = \sin(m\pi x_1)\sin(n\pi x_2)$$
$$\lambda_{m,n} = \pi^2(m^2 + n^2)$$

- For convenience, set $\mathcal{Y} = 10$, $s = 0.25$, and use 1089 DOFs.
- Strong scaling - $1,024,000$ eigenpairs.
- Weak scaling - $256,000 \times N_{\text{procs}}$ eigenpairs.

Strong Scaling Performance

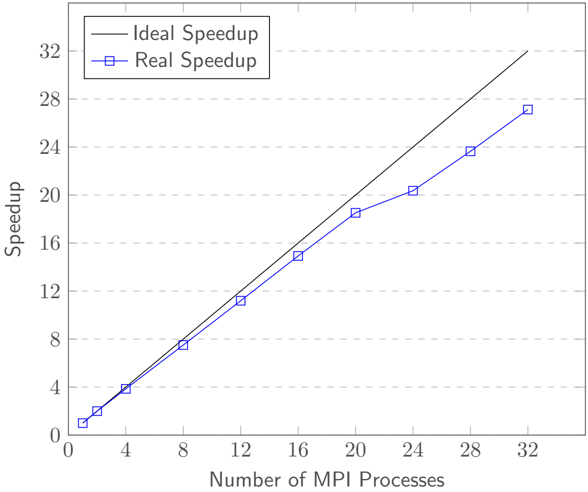


Figure: Strong scaling performance in a simple test case.

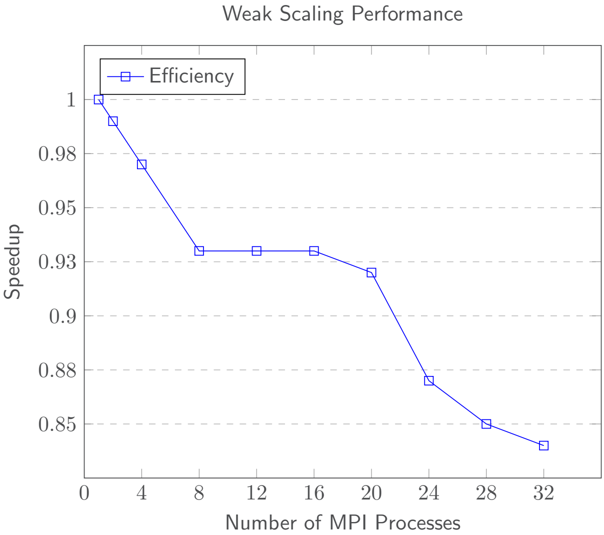


Figure: Weak scaling performance in a simple test case.

Conclusions

- Reviewed spectral Fractional Laplacian and previous research.
- Introduced an analytical extension to the diagonalization technique.
- Showed relation to a quadrature scheme for the Balakrishnan formulation.
- Presented Error analysis of the method in $L^2(\Omega)$.
- Demonstrated theoretical convergence rate with numerical examples.
- Demonstrated parallel performance of the algorithm.

Continuing (and Future) Work

- Refine the parallel performance on larger meshes.
  - Use linear solvers in PETSc/Trilinos.
  - Scale on cluster with multiple nodes, instead of just across cores.
- Develop more robust code for 3D.
  - Matrix free method.
  - Use threads to parallelize matrix solve.
- Derive error analysis in natural norm $\mathbb{H}^s(\Omega)$.
- Get estimates when $\partial\Omega$ or f are "rougher."

Thank you for your attention!

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Some clarifications.

$$A_k = \frac{\sqrt{2}\cos(\pi s)}{\mu_k^{s/2}\mathcal{Y}J_{1-s}(\eta_k)}\,.$$

The digamma function:

$$\psi(z) = \frac{d}{dz}\ln\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)}\,.$$

$$\psi'(z) = \frac{d^2}{dz^2}\ln\Gamma(z) = \sum_{n=0}^{\infty}\frac{1}{(z+n)^2}\,.$$

For large arguments:

$$\psi'(x) \sim \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \cdots \lesssim \frac{1}{x}\,.$$

Hurwitz Zeta function:

$$\zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+s)^a}\,.$$