# An Analytical Diagonalization Technique for Approximating the Spectral Fractional Laplacian

Shane E. Sawyer    Abner J. Salgado

University of Tennessee, Knoxville

## Abstract

We study the spectral fractional Laplacian $(-\Delta)^s$ in a bounded domain $\Omega \subset \mathbb{R}^d$. As in previous works we use the Caffarelli-Silvestre extension to convert it into a Dirichlet-to-Neumann mapping in $\mathbb{R}^{d+1}_+$. A diagonalization scheme is used to reduce the computational complexity and expose the inherent parallelizability of the method.

We refine the diagonalization scheme by proposing an analytic approach to compute the eigenpairs of the eigenvalue problem in the extended dimensions, avoiding the numerical instability in approximating the eigenpairs with a finite element method. We demonstrate that this new analytical approach is related to certain quadrature schemes used to approximate the spectral fractional Laplacian. We further show that this novel algorithm maintains exponential convergence. Numerical examples in two dimensions demonstrate the performance of the method.

## Problem Statement

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain. Given $s \in (0,1)$ and a sufficiently smooth $f$, find $u$ such that

$$(-\Delta)^s u = f \quad \text{in } \Omega .$$

Instead of solving the above problem, use the Caffarelli-Silvestre extension technique [3, 6] to reformulate the problem. Set $\mathcal{C} = \Omega \times (0,\infty)$ and find $\mathcal{U}$ such that

$$\operatorname{div}(y^\alpha \nabla \mathcal{U}) = 0 \text{ in } \mathcal{C}, \qquad \mathcal{U} = 0 \text{ on } \partial_L \mathcal{C}, \qquad \frac{\partial \mathcal{U}}{\partial \nu^\alpha} = d_s f \text{ on } \Omega \times \{0\}, \tag{1}$$

where $\alpha = 1 - 2s$, $\partial_L \mathcal{C}$ denotes the lateral boundary, $\frac{\partial}{\partial \nu^\alpha}$ is the co-normal derivative, and $d_s = \frac{2^{1-2s}\Gamma(1-s)}{\Gamma(s)}$ is a normalizing constant.

**Note:** $u(\cdot) = \mathcal{U}(\cdot, 0)$.

## Weak Formulation

- First, truncate the cylinder to $\mathcal{C}_\mathcal{Y} = \Omega \times (0, \mathcal{Y})$.
- Introduce *weighted* $L^2$ space:

$$\|w\|^2_{L^2(y^\alpha, \mathcal{C}_\mathcal{Y})} := \int_{\mathcal{C}_\mathcal{Y}} y^\alpha w(x)^2 dx\, dy < \infty .$$

- Define *weighted* Sobolev space:

$$\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}) = \{ w \in L^2(y^\alpha, \mathcal{C}_\mathcal{Y}) : \nabla w \in L^2(y^\alpha, \mathcal{C}_\mathcal{Y}),\, w|_{\partial_L \mathcal{C}_\mathcal{Y}} = 0 \} .$$

- The problem is now: find $\mathcal{U}_\mathcal{Y} \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y})$ such that

$$\int_{\mathcal{C}_\mathcal{Y}} y^\alpha (\nabla \mathcal{U}_\mathcal{Y} \cdot \nabla v) dx\, dy = a_{\mathcal{C}_\mathcal{Y}}(\mathcal{U}_\mathcal{Y}, v) = d_s \langle f, \operatorname{tr} v \rangle , \qquad \forall v \in \mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}) .$$

## Tensorial Finite Element Method

- Observe: $\mathring{H}^1_L(y^\alpha, \mathcal{C}_\mathcal{Y}) = H^1_0(\Omega) \otimes H^1_\mathcal{Y}(y^\alpha, (0, \mathcal{Y}))$.
- Previous work [5, 1] constructed a FE discretization by

$$\mathbb{V}_h \otimes \mathcal{S}_\mathcal{Y} \subset H^1_0(\Omega) \otimes H^1_\mathcal{Y}(y^\alpha, (0, \mathcal{Y}))$$

  - $\mathcal{T}_h$ is shape regular, conforming mesh of $\Omega$ and $\mathbb{V}_h$ is the space of piecewise linear elements over $\mathcal{T}_h$
  - $\mathcal{S}_\mathcal{Y}$ is a FE discretization of the extended dimension
- Solution representation:

$$\mathcal{U}^\mathcal{K}_{h,\mathcal{Y}}(x,y) = \sum_{k=1}^{\mathcal{K}} U_k(x) v_k(y) , \quad U_k \in \mathbb{V}_h ,\, v_k \in \mathcal{S}_\mathcal{Y} .$$

- Optimal convergence rate required *hp*-FEM in the extended dimension [1].

## Diagonalization Method

- Substitute solution into weak form:

$$a_{\mathcal{C}_\mathcal{Y}}(\mathcal{U}^\mathcal{K}_{h,\mathcal{Y}}, V\Psi) = d_s \int_\Omega f(x) V(x) \Psi(0)\, dx .$$

- Simplify by: Find $(\psi, \mu) \in \mathcal{S}_\mathcal{Y} \setminus \{0\} \times \mathbb{R}$ such that

$$\int_0^\mathcal{Y} y^\alpha \psi'(y) w'(y)\, dy = \mu \int_0^\mathcal{Y} \psi(y) w(y)\, dy , \quad \forall w \in \mathcal{S}_\mathcal{Y} .$$

- Now the problem is: For $k = 1, 2, \ldots, \mathcal{K}$, find $U_k \in \mathbb{V}_h$ such that

$$\int_\Omega \nabla U_k(x) \cdot \nabla V(x)\, dx + \mu_k \int_\Omega U_k(x) V(x)\, dx = d_s \psi_k(0) \langle f, V \rangle , \quad \forall V \in \mathbb{V}_h .$$

- **Issue:** computing eigenvalue problem is unstable [7].
- Instead, solve eigenvalue problem exactly.

$$\psi''(y) + \frac{\alpha}{y} \psi'(y) + \mu \psi(y) = 0 , \quad y \in (0, \mathcal{Y}) .$$

- Solutions are

$$\mu_k^{(\frac{1}{2})} = \left( \frac{(k - \frac{1}{2})\pi}{\mathcal{Y}} \right)^2 , \quad \psi_k^{(\frac{1}{2})}(0) = \sqrt{\frac{2}{\mathcal{Y}}} , \quad \mu_k^{(s)} = \left( \frac{\eta_k}{\mathcal{Y}} \right)^2 , \quad \psi_k^{(s)}(0) = \frac{2^{s+1/2}}{\mu_k^{s/2} \mathcal{Y} J_{1-s}(\eta_k) \Gamma(1-s)} . \tag{2}$$

## Quadrature and Error Analysis

- Approximate solution may be written as

$$u^\mathcal{K}_{h,\mathcal{Y}} = d_s \sum_{k=1}^{\mathcal{K}} |\psi_k(0)|^2 (\mu_k I - \Delta_h)^{-1} P_h f .$$

- Interpret this as a quadrature of the Balakrishnan formula [2]

$$(-\Delta_h)^{-s} = \frac{2\sin(s\pi)}{\pi} \int_0^\infty t^\alpha (t^2 I - \Delta_h)^{-1}\, dt .$$

- Error is controlled via

$$\|u - u^\mathcal{K}_{h,\mathcal{Y}}\|_{L^2(\Omega)} \le \|u - u_h\|_{L^2(\Omega)} + \|u_h - u^\infty_{h,\mathcal{Y}}\|_{L^2(\Omega)} + \|u^\infty_{h,\mathcal{Y}} - u^\mathcal{K}_{h,\mathcal{Y}}\|_{L^2(\Omega)}$$
$$\lesssim \text{Discretization error} + \text{Quadrature error} + \text{Truncation error}$$

- Discretization error is controlled by $h^{2s}$ [4].
- For $s = \frac{1}{2}$, we have

$$\|u - u^\mathcal{K}_{h,\mathcal{Y}}\|_{L^2(\Omega)} \lesssim h^{2s} + \frac{\exp(-\mathcal{Y}\sqrt{\lambda_1})}{\sqrt{\lambda_1}} + \frac{\mathcal{Y}}{\mathcal{K}} ,$$

  where $\lambda_1$ is the principal eigenvalue of the Laplacian.
- For $s \neq \frac{1}{2}$, we have

$$\|u - u^\mathcal{K}_{h,\mathcal{Y}}\|_{L^2(\Omega)} \lesssim h^{2s} + \exp(-\mathcal{Y}/\sqrt{\lambda_1}) + \left( \frac{\eta_k}{\mathcal{Y}} \right)^{-2s+\varepsilon} ,$$

  for all $\varepsilon > 0$ such that $0 < 2s - \varepsilon$ and $\lambda_1$ is the principal eigenvalue of the Laplacian.

## Numerical Recipe

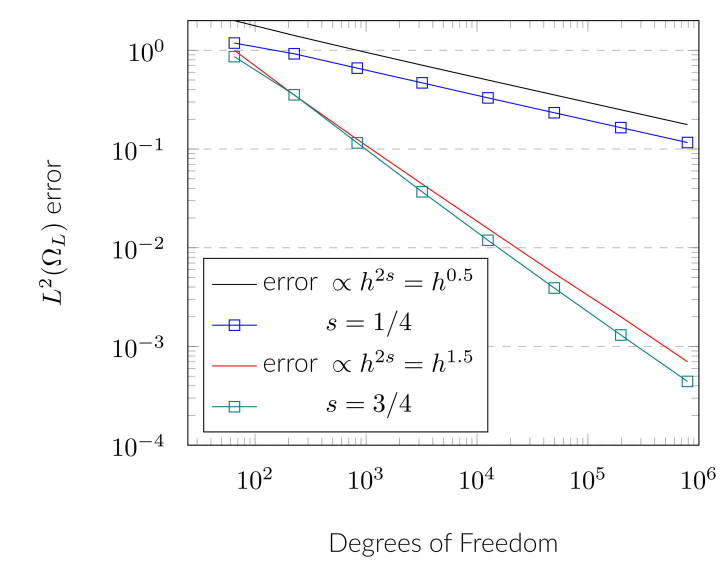
To achieve theoretical convergence for a given $h$, we require

$$\mathcal{Y} = 2s|\log(h)| \quad \text{and} \quad \mathcal{K} = \frac{2s|\log(h)|}{h} .$$

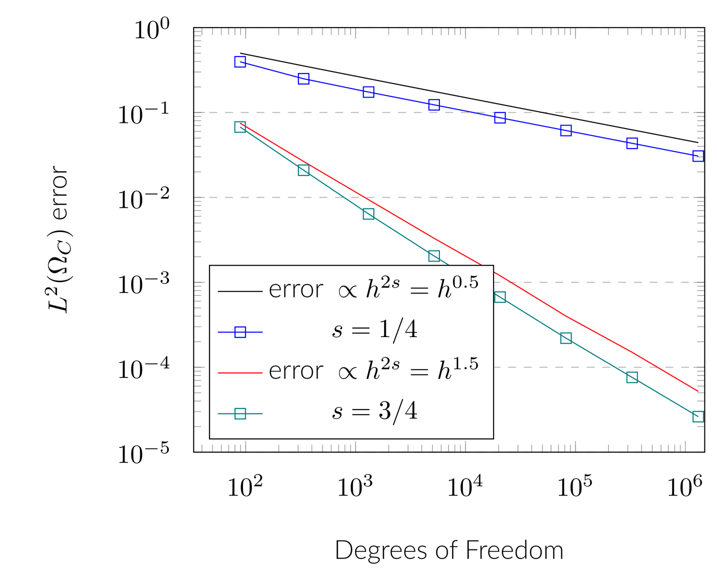
## Computational Methodology

- Implemented in deal.II (9.4.2-r3)
- Parallelized with OpenMPI (4.1.5)
- Single node Intel Xeon Gold 6246R
- Fedora release 36

## Numerical Results

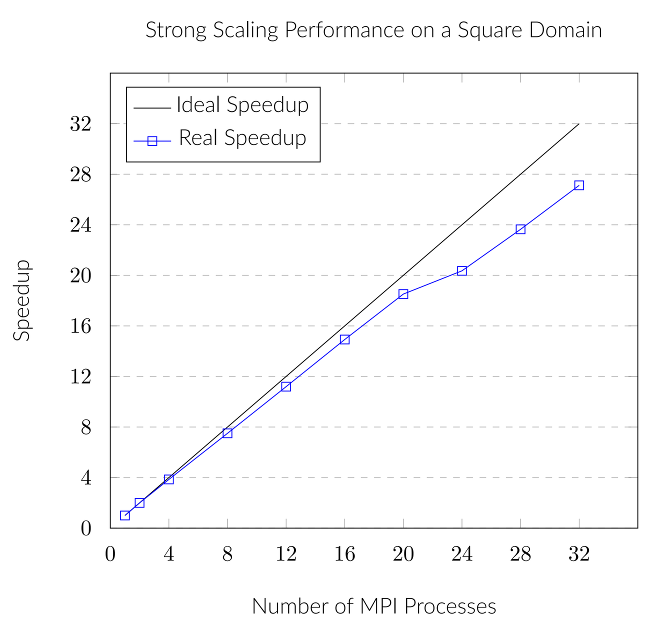


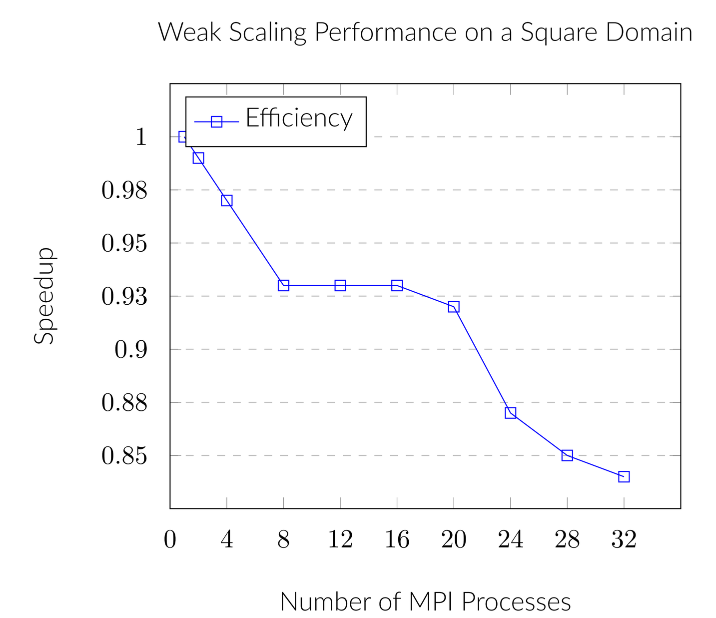

(a) $\Omega_L = (0,1)^2$ with linear combination of sine functions as exact solution.




(b) $\Omega_C = \{(x,y) : x^2 + y^2 \le 1\}$ and exact solution is a scaled Bessel function of the first kind.

## Parallel Performance

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